Detailed Syllabus

Semester VI

Vector Spaces and Metric Spaces

Sem Exam: 80

Course No. UGM-601 **Sess. Exam:** 20

Unit-I

Definition and examples of vector spaces, subspaces of a vector space and the quotient space, Linearly dependence and linearly independence of a set of vectors, Linear span, Excercises and examples based on these concepts.

Unit-II

Basis and dimension of a vector space, isomorphic vector spaces, finite and infinite dimensional vector spaces with plenty of examples, Dual space of a finite dimensional vector space-definition and examples, Dimension of dual space, Exercises based on these concepts.

Unit-III

Linear Transformations on vector space and their examples, algebra of linear transformations on a vector space, matrix representation of a linear transformation on finite dimensional vector spaces. Kernel and range of a linear transformation, inverse of linear transformation on finite dimensional vector spaces. Examples and exercises based on these concepts.

Unit-IV

Denumerable and non-Denumerable sets and their examples. Open set and closed set on the real line, their examples and properties. Limit point of a set.

Heine Borel Theorem for closed and bounded intervals. Bolzano Weirstrass Theorem. Examples and ecercises based on these concepts.

Unit-V

Definition of metric space with examples. Open sets and closed sets in metric space. Interior, closure and boundary of a set in metric spaces. Equivalent conditions for open sets and closed sets. Convergence of sequences. Continuous maps and their characterisations. Examples and excercises based on these concepts.

Reference Text Books: (i) N.S. Gopalakrishnan, *University Algebra*, New Age International (P) Limited, Publishers.

(ii) Kenneth Hoffman, Ray Kunze, *Linear Algebra*,Prentice Hall India.

(iii) Singh, S. and Zameerudin, Q.,*Modern Algebra*, Vikas Publishing House Pvt.ltd.

(iv) Shanti Naryanan, M. D. Rai Singhania, *Elements of Real Analysis*, S. Chand Publishing House Pvt. Ltd.

Note (i) The question paper shall consist of ten questions, two from each unit. The students will require to do any five questions selecting one from each each unit.

(ii) Internal assessment will be of 20 marks for the two written assignments 10 marks each.

Table 1: **Contents**

Unit-I

Lesson-I **Vector Spaces**

1.0 **Structure**

1.1 Introduction

1.2 Objectives

1.3 Vector spaces

1.3.1 Definition of vector space

1.3.2 Theorem

1.4 Examples

1.5 Let Us Sum Up

1.6 Lesson End Exercise

1.7 University Model Questions

1.8 Suggested Readings

(1.1) Introduction: As we are familiar with the notion of Rings and fields where a non empty set carries two binary operations namely addition (+) and multiplication (.). Similarly here in this lesson we are going to introduce a notion of **vector space**. A **vector space** is a non empty set of vectors with two operations (one is *internal binary operation among the vectors and another is external operation on elements of set of vectors and elements of field known as scalar multiplication).*

(1.2) Objectives: *(i) students will get understanding of a set of vectors.*

(ii) through this lesson students will understand the relation between algebra and geometry.

(1.3) Vector Spaces

(1.3.1) Definition: *A non empty set V is said to be a vector space over a field F under a binary operation $+$ and scalar multiplication $\lambda : F \times V \to V$*

defined by $\lambda(\alpha, v) = \alpha v$ *if the following properties are satisfied:*

I) $(V, +)$ *is an abelian group.*

II) properties under scalar multiplication

(1) $(\alpha + \beta)v = \alpha v + \beta v, \ \forall \ \alpha, \ \beta \ \in F$ *and* $\forall \ v \ \in V$

(2) $\alpha(u + v) = \alpha u + \alpha v, \ \ \forall \ \ u \ , v \ \in V$ *and* $\forall \ \alpha \ \in F$

(3) $\alpha(\beta v) = (\alpha\beta)v, \ \forall \ \alpha, \beta \in F$ *and* $\forall v \in V$

(4) $1v = v, \ \forall v \ \in V.$

It is generally denoted as $V(F)$.

(1.3.2) Theorem*: Let* $V(F)$ *be a vector space over* F. *Then*

(i) $0v = 0, \forall \ v \ \in \ V$ *(ii)* $\alpha 0 = 0, \forall \ \alpha \ \in \ F.$

(iii) $\alpha(-v) = -\alpha v = (-\alpha)v$ *(iv)* $\alpha(u - v) = \alpha u - \alpha v, \forall \ \alpha \ \in \ F$ *and* $\forall \ u, v \ \in \ V.$

(v) $\alpha v = 0$ *if and only if either* $\alpha = 0$ *or* $v = 0$.

Proof*:(i) Since* $0 + 0 = 0$ *in* F.

Therefore, $(0 + 0)v = 0v + 0v, \forall \ v \ \in \ V \Rightarrow 0v = 0v + 0v$

$\Rightarrow 0 + 0v = 0v + 0v \Rightarrow 0 = 0v.$

(ii) Since $0 + 0 = 0$ *in* V.

Therefore, $\alpha(0 + 0) = \alpha 0 + \alpha 0, \forall \ \alpha \ \in \ F \Rightarrow \alpha 0 = \alpha 0 + \alpha 0$

$\Rightarrow \alpha 0 + 0 = \alpha 0 + \alpha 0 \Rightarrow 0 = \alpha 0.$

(iii) Since $\alpha + (-\alpha) = 0 \Rightarrow (\alpha + (-\alpha))v = 0v = 0$

$\Rightarrow \alpha v + (-\alpha)v = 0 \Rightarrow (-\alpha)v = -(\alpha v).$

Similarly, $\alpha(-v) = -\alpha v.$

(iv) By the property of vector space, we have $\alpha[u + (-v)] = \alpha u + \alpha(-v) = \alpha u - \alpha v$ *(by (iii)).*

(v) Suppose that $\alpha v = 0$ *and* $\alpha \neq 0$. *Then there exists* $\alpha^{-1} \ \in \ F$. *This implies that* $\alpha^{-1}(\alpha v) = \alpha^{-1}0 = 0$ *(using property (i))*

$\Rightarrow (\alpha^{-1}\alpha)v = 0 \Rightarrow 1v = 0 \Rightarrow v = 0.$

Conversely if either $\alpha = 0$ or $v = 0$. Then in any of cases $\alpha v = 0$.

(1.4) Examples

1. *Let $V = \mathbb{R}^n$. Define operation $+$ and scalar multiplication on V as $(x_1, x_2, \ldots, x_n)+(y_1, y_2, \ldots, y_n) = (x_1+y_1, \ldots, x_n+y_n)$ and $\alpha(x_1, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n), \forall\ \alpha \in F$ and for all $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in \mathbb{R}^n$ respectively. Then V is a vector space over $\mathbb{R}$.*

For $n = 1, V = \mathbb{R}$, is a vector space over itself. For $n = 2, V = \mathbb{R}^2 = \{(x, y)|x, y \in \mathbb{R}\}$ is a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication.

Properties under +

(A_1) *Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be any two elements of $\mathbb{R}^n$. Then $x+y = (x_1+y_1, \ldots, x_n+y_n) \in \mathbb{R}^n$ because $x_i+y_i \in \mathbb{R}, \forall\ i$. This implies that V is closed under $+$.*

(A_2) *Let x, y and z be any elements of V. Then $x + (y + z) = (x_1, \ldots, x_n) + (y_1 + z_1, \ldots, y_n + z_n)$*

$$
\begin{aligned}
&= (x_1 + (y_1 + z_1), \ldots, x_n + (y_n + z_n)) \\
&= ((x_1 + y_1) + z_1, \ldots, (x_n + y_n) + z_n) \\
&= (x_1 + y_1, \ldots, x_n + y_n) + (z_1, z_2, \ldots, z_n) \\
&= (x + y) + z
\end{aligned}
$$

(A_3) *There exists $\mathbf{0} = (0, \ldots, 0) \in V$ such that $x + \mathbf{0} = (x_1, \ldots, x_n) + (0, \ldots, 0)$*

$$
\begin{aligned}
&= (x_1 + 0, \ldots, x_n + 0) \\
&= (x_1, \ldots, x_n) = x, \ \forall\ x \in V
\end{aligned}
$$

(A_4) *For each*

$$x = (x_1, \ldots, x_n) \ \in \ V$$

there exists $-x = (-x_1, \ldots, -x_n) \in V$

such that $x + (-x) = (x_1 - x_1, \ldots, x_n - x_n) = (0, \ldots, 0) = \mathbf{0}$

(A_5) $x + y = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) = (y_1 + x_1, \ldots, y_n + x_n) = y + x.$

Properties under scalar multiplication: *Let* α, β *be scalars and* $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_n)$ *be any two elements of* $\mathbb{R}^n$. *Then*

(S_1) $\alpha(x + y) = \alpha(x_1 + y_1, \ldots, x_n + y_n)$

$$= (\alpha(x_1 + y_1), \ldots, \alpha(x_n + y_n))$$

$$= (\alpha x_1 + \alpha y_1, \ldots, \alpha x_n + \alpha y_n)$$

$$= (\alpha x_1 +, \ldots, \alpha x_n) + (\alpha y_1, \ldots, \alpha y_n)$$

$$= \alpha x + \alpha y$$

$\Rightarrow \alpha(x + y) = \alpha x + \alpha y.$

(S_2) $(\alpha + \beta)x = (\alpha + \beta)(x_1, \ldots, x_n)$

$$= ((\alpha + \beta)x_1, \ldots, (\alpha + \beta)x_n)$$

$$= (\alpha x_1 + \beta x_1, \ldots, \alpha x_n + \beta x_n)$$

$$= (\alpha x_1, \ldots, \alpha x_n) + (\beta x_1, \ldots, \beta x_n)$$

$$= \alpha x + \beta x$$

$\Rightarrow (\alpha + \beta)x = \alpha x + \beta x.$

(S_3) $(\alpha\beta)x = (\alpha\beta)(x_1, \ldots, x_n)$

$$= ((\alpha\beta)x_1, \ldots, (\alpha\beta)x_n)$$

$$= (\alpha(\beta x_1), \ldots, \alpha(\beta x_n))$$

$$= \alpha(\beta x_1, \ldots, \beta x_n)$$

$$= \alpha(\beta x)$$

$\Rightarrow (\alpha\beta)x = \alpha(\beta x)$

$(S4)$ $1x = 1(x_1, \ldots, x_n)$

$$= (x_1, \ldots, x_n) = x.$$

2. *Let* $V = \mathbb{R}^2 = \{(x, y)|x, y \in \mathbb{R}\}$. *Define addition and scalar multiplication on* V *as* $(x_1, y_1) + (x_2, y_2) = (x_1 + y_1, x_2 + y_2)$ *for all* $x_1, x_2, y_1, y_2 \in \mathbb{R}$ *and* $\alpha(x_1, x_2) = (\alpha x_1, 0) \ \ \forall \alpha \in \mathbb{R}$ *and* $\forall (x_1, x_2) \in V$ *is not a vector space.*

It is easy to see that $(V, +)$ *is an abelian group. The property* $1(x_1, x_2) = (x_1, 0) \neq (x_1, x_2)$ *which shows that* V *is not a vector space.*

3. *Let* $V = \{a_0 + a_2 x + \ldots a_n x^n | a_0, a_1, \ldots a_n \in F\}$ *be a set of polynomials over a field* F. *Then* V *is a vector space over* F *under the operation addition* $+$ *and scalar multiplication defined as* $(a_0 + a_1 x + \ldots a_n x^n) + (b_0 + b_1 x + \ldots b_m x^m) = (a_0 + b_0) + (a_1 + b_1)x + \ldots (a_m + b_m)x^m + \ldots a_n x^n$, *if* $m < n$ *and* $\alpha(a_0 + a_1 x + \ldots + a_n x^n) = \alpha a_0 + \alpha a_1 x + \ldots + \alpha a_n x^n$ *respectively.*

4. *Let* $V_n = \{a_0 + a_2 x + \ldots a_n x^n | a_0, a_1, \ldots a_n \in F\}$ *be a set of polynomials over a field* F *with* $\deg f(x) \leq n$, *for all* $f(x) \in V_n$. *Then* V_n *is a vector space.*

Properties under +: (A_1). *Let* $f(x), g(x)$ *be any elements of* V_n. *Then* $\deg(f(x)) \leq n$ *and* $\deg(g(x)) \leq n$. *Now, We know that* $\deg(f(x) + g(x)) \leq \max\{\deg(f(x)), \deg(g(x))\} \leq n$. *This implies that* $f(x) + g(x \in V_n$.

A_2. *Since* $\deg 0 = -\infty$, *so* $0 \in V$ *such that* $f(x) + 0 = 0 + f(x) = f(x), \forall f(x) \in V_n$.

A_3. *Let* $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots, g(x) = b_0 + b_1 x + b_2 x^2 + \ldots$ *and* $h(x) = c_0 + c_1 x + c_2 x^2 + \ldots$ *be any elements of* V_n. *Then it is easy to see that* $(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$ *and* $f(x) + g(x) = g(x) + f(x)$. *Therefore,* V_n *is an abelian group.*

Properties under scalar multiplication:

(S_1) *Let* $\alpha \in F$ *and* $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$,$g(x) = b_0 + b_1 x + b_2 x^2 + \ldots$ *in* V_n. *Then* $\alpha(f(x) + g(x)) = \alpha((a_0 + a_1 x + a_2 x^2 + \ldots) + (b_0 + b_1 x + b_2 x^2 + \ldots))$

$$= \alpha(a_0 + b_0) + \alpha(a_1 + b_1)x + \ldots$$

$$= (\alpha a_0 + \alpha a_1 x + \ldots) + (\alpha b_0 + \alpha b_1 x + \ldots)$$

$$= \alpha f(x) + \alpha g(x)$$

(S_2) *Let* $\alpha,\ \beta \in F$ *and* $f(x) = a_0 + a_1x + a_2x^2 + \ldots \in V_n$.

Then $(\alpha+\beta)f(x) = (\alpha+\beta)a_0 + (\alpha+\beta)a_1x + \ldots$

$$= (\alpha a_0 + \beta a_0) + (\alpha a_1x + \beta a_1x) + \ldots$$

$$= (\alpha a_0 + \alpha a_1x + \ldots + \beta a_0 + \beta a_1x + \ldots))$$

$$= \alpha f(x) + \beta f(x)$$

(S_3) *Let* $\alpha,\ \beta \in F$ *and* $f(x) = a_0 + a_1x + a_2x^2 + \ldots \in V_n$.

Then $(\alpha\beta)f(x) = (\alpha\beta)a_0 + (\alpha\beta)a_1x + \ldots$

$$= \alpha(\beta a_0) + \alpha(\beta a_1)x + \ldots$$

$$= \alpha(\beta f(x))$$

(S_4) $1f(x) = f(x),\ \forall f(x) \in V_n$.

Therefore, V_n *is a vector space over* F.

5. *Let* V *be the abelian group of positive real numbers for multiplication. Define scalar multiplication in* V *by* $ax = x^a$, $a \in \mathbb{R}$ *and* $x \in V$. *Then* V *is a vector space over* $\mathbb{R}$.

Solution *It is enough to verify the properties under scalar multiplication:*

(1) *Let* $a,\ b \in \mathbb{R}$ *and* $x \in V$,

then $(a+b)x = x^{a+b}$

$$= x^a x^b$$

$= (ax)(bx) = ax + bx$ *(because here + in* V *means multiplication).*

(2) *Let* $a \in \mathbb{R}$ *and* $x,\ y \in V$.

Then $a(xy) = (xy)^a$

$$= x^a y^a$$

$$= (ax)(ay)$$

(3) *Let* $a,\ b \in \mathbb{R}$ *and* $x \in V$.

Then $(ab)x = x^{ab}$

$= (x^b)^a$

$= (bx)^a$

$= a(bx)$

(4) *Let* $1 \in \mathbb{R}$ *and* $x \in V$*. Then* $1x = x^1 = x$.

6. *Let* F *be a field and* $M_{m\times n}(F)$ *be the set of all* $m \times n$ *matrices over* F*. Then* $M_{m\times n}(F)$ *is a vector space over* F *under the addition of matrices and multiplication of matrix by a scalar as internal and external operations on* $M_{m\times n(F)}$ *respectively.*

Solution:*Let* $V = M_{m\times n}(F)$.

Properties under $+$:

(A_1) *Let* $A = [a_{ij}]$ *and* $B = [b_{ij}]$ *be two matrices of order* $m \times n$ *over a field* F*. Then* $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$.

This implies that $A + B \in V$.

(A_2) *Let* $A = [a_{ij}]$, $B = [b_{ij}]$ *and* $C = [c_{ij}]$ *be any elements of* V.

Then $(A + B) + C = [a_{ij} + b_{ij}] + [c_{ij}]$

$= [(a_{ij} + b_{ij}) + c_{ij}]$

$= [a_{ij} + (b_{ij} + c_{ij})]$

$= [a_{ij}] + [b_{ij} + c_{ij}]$

$= A + (B + C)$.

(A_3) *There exists* $O = [0]_{m\times n} \in V$*, where* 0 *is an identity element of* F *such that* $A + O = [a_{ij} + 0] = [a_{ij}] = A$.

(A_4) *For each* $A \in V$ *there exists* $-A \in V$ *such that* $A+(-A) = [a_{ij} - a_{ij}] = [0]_{m\times n}$.

(A_5) *Let* $A = [a_{ij}]$ *and* $B = [b_{ij}]$ *be any two elements of* V*. Then* $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = [b_{ij}] + [a_{ij}] = B + A$.

Properties under scalar multiplication:

(S_1) *Let* $\alpha \in F, \beta \in F$ *and* $A \in V$. *Then* $(\alpha+\beta)A = [(\alpha+\beta)a_{ij}] = [\alpha a_{ij} + \beta a_{ij}] = [\alpha a_{ij}] + [\beta a_{ij}] = \alpha[a_{ij}] + \beta[a_{ij}] = \alpha A + \beta A$.

(S_2) *Let* $\alpha \in F$ *and* $A, B \in V$. *Then* $\alpha(A+B) = \alpha([a_{ij}] + [b_{ij}]) = \alpha\ [a_{ij} + b_{ij}] = [\alpha(a_{ij} + b_{ij})] = [\alpha a_{ij} + \alpha b_{ij}] = [\alpha a_{ij}] + [\alpha b_{ij}] = \alpha A + \alpha B$.

(S_3) $(\alpha\beta)A = (\alpha\beta)[a_{ij}] = [(\alpha\beta)a_{ij}] = [\alpha(\beta a_{ij})] = \alpha[\beta a_{ij}] = \alpha(\beta[a_{ij}] = \alpha\beta A)$.

(S_4) $1A = 1[a_{ij}] = [1a_{ij}] = [a_{ij}] = A$.

7. *Let* $V = F^S$ *be the set of all functions from a non-empty set* S *to field* F. *Then* V *is a vector space over* F *under operations + (sum of functions) and scalar multiplication defined by* $(f+g)(s) = f(s) + g(s), \forall\ s \in S$ *and* $(cf)(s) = cf(s), \forall\ c \in F$ *and* $\forall\ s \in S$ *respectively.*

Solution:

Properties under +:

(A_1) *Let* $f \in V$ *and* $g \in V$. *Since* $f(s) + g(s) \in F$,

so $(f+g)(s) = f(s) + g(s) \in F$ *which implies that* $f + g \in V$.

(A_2) *Let* f, g, h *be any elements of* V.

Then $[(f+g)+h](s) = (f+g)(s) + h(s) = [f(s) + g(s)] + h(s)$

$$= f(s) + [g(s) + h(s)]$$

$$= f(s) + (g+h)(s)$$

$$= [f + (g+h)](s)$$

$\Rightarrow (f+g) + h = f + (g+h)$.

(A_3) *Define a function* $O : S \to F$ *by* $O(s) = 0,\ \forall\ s \in S$.

Then $O \in V$ *and* $(f+O)(s) = f(s) + O(s) = f(s) + 0 = f(s)$ *which implies that* $f + O = f,\ \forall\ f \in V$.

(A_4) *For each* $f \in V$, *define a function* $-f : S \to F$ *define by* $(-f)(s) = -f(s), \forall\ s \in V$.

Then $(-f+f)(s) = -f(s) + f(s) = 0 = O(s)$

$\Rightarrow -f + f = O.$

(A_5) $(f+g)(s) = f(s) + g(s) = g(s) + f(s) = (g+f)(s)$

$\Rightarrow f + g = g + f.$

Properties under scalar multiplication*:*

(S_1) *Since* $cf(s) \in F, \; \forall\, c \in F$ *and* $\forall\, f \in V.$

Therefore $cf \in V \; \forall\, c \in F$ *and* $\forall\, f \in V.$

(S_2) *Let* $c_1, c_2 \in F$ *and* $f \in V$. *Then* $[(c_1+c_2)f](s) = (c_1+c_2)f(s)$

$$= c_1 f(s) + c_2 f(s)$$

$$= (c_1 f + c_2 f)(s)$$

$\Rightarrow (c_1+c_2)f = (c_1 f + c_2 f).$

(S_3) *Let* $c \in F$ *and* $f_1, f_2 \in V$.

Then $c(f_1+f_2)(s) = c[f_1(s) + f_2(s)]$

$$= cf_1(s) + cf_2(s)$$

$$= (cf_1)(s) + (cf_2)(s)$$

$$= [cf_1 + cf_2](s)$$

$\Rightarrow c(f_1+f_2) = cf_1 + cf_2.$

(S_4) $[(c_1c_2)f](s) = (c_1c_2)f(s)$

$$= c_1(c_2 f(s))$$

$$= c_1[(c_2 f)(s)]$$

$$= [c_1(c_2 f)](s)$$

$\Rightarrow (c_1c_2)f = c_1(c_2 f).$

(S_5) $(1f)(s) = 1f(s) = f(s) \Rightarrow 1f = f.$

(1.6) Lesson End Excercise

1. *If* F *is a field, verify that* F^n *is a vector space over the field* F *under the operations addition (+) and scalar multiplication defined as* $(x_1, \ldots, x_n) +$

$(y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$ *and* $\alpha(x_1, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n)$ *respectively.*

2. *Let* V *be set of all real valued continuous functions defined in closed interval* $[a, b]$. *Then show that* V *is a vector space over* $\mathbb{R}$ *with addition and scalar multiplication defined by* $(f+g)(x) = f(x) + g(x), \forall\ f, g \in V$ *and* $(\alpha f)(x) = \alpha f(x), \forall\ \alpha \in \mathbb{R}, f \in V$.

3. *Let* V *set of all real valued continuous functions defined on* $[0, 1]$ *such that* $f(\frac{2}{3}) = 2$. *Show that* V *is not a vector space over* $\mathbb{R}$ *under addition and scalar multiplication defined as* $(f+g)(x) = f(x) + g(x), \forall\ f, g \in V$ *and* $(\alpha f)(x) = \alpha f(x), \forall\ \alpha \in \mathbb{R}, f \in V$.

Hint: *let* $f \in V$ *and* $g \in V$. *Then* $f(\frac{2}{3}) = 2$ *and* $g(\frac{2}{3}) = 2$. *But* $(f+g)(\frac{2}{3}) = f(\frac{2}{3}) + g(\frac{2}{3}) = 2 + 2 = 4 \neq 2$. *So* V *is not closed under addition* $(+)$.

4. *Show that the set* $V = \{f : \mathbb{R} \to \mathbb{R} \mid \frac{d^2 f}{dx^2} + 3\frac{df}{dx} = 0\ \}$ *is a vector space over* $\mathbb{R}$ *under the operations as defined in exercise* 3.

5. *Let* V *be a vector space over the field of numbers* $\mathbb{R}$ *and* $W = \{(u, v) :\ u, v \in V\}$. *Define addition in* W *co-ordinate wise and scalar multiplication in* W *by* $(a + \iota b)(u, v) = (au - bv, av + bu)$, $a, b \in \mathbb{R}$, $\iota = \sqrt{-1}$. *Show that* W *is a vector space over* $\mathbb{C}$.

Solution: Properties under addition

(A_1) *Let* $(v_1, v_2) \in V$ *and* $(u_1, u_2) \in V$. *Then* $(v_1, v_2) + (u_1, u_2) = (v_1 + u_1, v_2 + u_2) \in V$. *Therefore* V *is closed under addition.*

(A_2) *Let* $v = (v_1, v_2) \in V$, $w = (w_1, w_2) \in V$ *and* $u = (u_1, u_2) \in V$.

Then $(v + w) + u = (v_1 + w_1, v_2 + w_2) + (u_1, u_2)$

$$= ((v_1 + w_1) + u_1, (v_2 + w_2) + u_2)$$

$$= (v_1 + (w_1 + u_1), v_2 + (w_2 + u_2))$$

$= (v_1,\, v_2) + ((w_1 + u_1),\, (w_2 + u_2))$

$= v + (w + u)$

$\Rightarrow (v + w) + u = v + (w + u).$

(A_3) *There exists* $(0,\, 0) \in V$ *such that* $(v_1,\, v_2) + (0,\, 0) = (v_1 + 0,\, v_2 + 0) = (v_1,\, v_2),\ \forall\ (v_1,\, v_2) \in V.$

(A_4) *For each* $v = (v_1,\, v_2)$*, there exists* $-v = (-v_1,\, -v_2)$ *such that* $v + (-v) = (v_1 - v_1,\, v_2 - v_2) = (0,\, 0).$

(A_5) *Let* $(v_1,\, v_2) \in V$ *and* $(u_1,\, u_2) \in V$*. Then* $(v_1,\, v_2) + (u_1,\, u_2) = (v_1 + u_1,\, v_2 + u_2) = (u_1 + v_1,\, u_2 + v_2) = (u_1,\, u_2) + (v_1,\, v_2).$

This shows that $(W, +)$ *is an abelian group under addition.*

Properties under Scalar multiplication*: Let* $z_1,\, z_2$ *be any two elements of* $\mathbb{C}$ *and* $v,\, u\ \in\ V$*. Then*

(S_1) $(z_1 + z_2)v = (a_1 + \iota b_1 + a_2 + \iota b_2)(v_1,\, v_2)$

$= ((a_1 + a_2) + \iota(b_1 + b_2))(v_1,\, v_2)$

$= ((a_1 + a_2)v_1 - (b_1 + b_2)v_2,\, (a_1 + a_2)v_2 + (b_1 + b_2)v_1)$

$= (a_1v_1 - b_1v_2,\, a_1v_2 + b_1v_1) + (a_2v_1 - b_2v_2,\, a_2v_2 + b_2v_1)$

$= (a_1 + \iota b_1)(v_1,\, v_2) + (a_2 + \iota b_2)(v_1,\, v_2)$

$= z_1(v_1,\, v_2) + z_2(v_1,\, v_2)$

$= z_1v + z_2v$

$\Rightarrow (z_1 + z_2)v = z_1v + z_2v.$

(S_2) $(a + \iota b)(v + u) = (a + \iota b)((v_1,\, v_2) + (u_1,\, u_2))$

$= (a + \iota b)(v_1 + u_1,\, v_2 + u_2)$

$= (a(v_1 + u_1) - b(v_2 + u_2),\, a(v_2 + u_2) + b(v_1 + u_1))$

$= (av_1 - bv_2,\, av_2 + bv_1) + (au_1 - bu_2,\, au_2 + bu_1)$

$= (a + \iota b)(v_1,\, v_2) + (a + \iota b)(u_1,\, u_2)$

$= (a + \iota b)v + (a + \iota b)u$

(S_3) $(z_1z_2)u = ((a_1 + \iota b_1)(a_2 + \iota b_2))(u_1,\, u_2)$

$= ((a_1a_2 - b_1b_2) + \iota(b_1a_2 + b_2a_1))(u_1,\, u_2)$

$= ((a_1a_2 - b_1b_2)u_1 - (b_1a_2 + b_2a_1)u_2,\, (a_1a_2 - b_1b_2)u_2 + (b_1a_2 + b_2a_1)u_1)$

$= (a_1(a_2u_1 - b_2u_2) - b_1(b_2u_1 + a_2u_2),\, a_1(a_2u_2 + b_2u_1) + b_1(a_2u_1 - b_2u_2))$

$= (a_1 + \iota b_1)(a_2u_1 - b_2u_2,\, b_2u_1 + a_2u_2)$

$= z_1(z_2(u_1,\, u_2))$

$= z_1(z_2u).$

(S_4) $1u = (1 + \iota 0)(u_1,\, u_2) = (u_1,\, u_2) = u.$

6. *Show that the set of all matrices of the form* $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ *where* $a,\, b \in \mathbb{C}$ *is a vector space over* $\mathbb{C}$ *under matrix addition and scalar multiplication.*

7. *Show that* $\mathbb{C}$ *is a vector space over field* $\mathbb{C}$.

8. *Show that every field* F *is a vector space over itself.*

Hint: *Since every field* F *is an abelian group under addition and scalar multiplication is the multiplication of elements of* F. *Therefore all properties of vector space are satisfied in* F.

9. *Show that* $\mathbb{R}$ *is not a vector space over* $\mathbb{C}$.

Hint: *Since* $\mathbb{R}$ *is not closed under scalar multiplication because* $\iota 3 = 3\iota$ *does not belong to* $\mathbb{R}$.

(1.7 University Model Questions)

1. *If* $(\mathbb{R},\, +,\, .)$ *be the field of real numbers, then show that* $\mathbb{R}$ *is a vector space over* $\mathbb{R}$.

2. *Define a vector space over a field. Let* $V = \{x \in \mathbb{R} | x > 0\}$. *For* $x,\, y \in V$, *let* $x \oplus y = xy$ *and for* $\alpha \in \mathbb{R}$ *and* $x \in V$, *let* $\alpha \odot x = x^\alpha$. *Prove that* V *is a vector space over* $\mathbb{R}$ *under the above operations.*

3. *Define vector space over a field F. For (x_1, x_2) and $(y_1, y_2) \in \mathbb{R}^2$, let $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and $\alpha(x_1, y_1) = (\alpha x_1, 0)$ for $\alpha \in \mathbb{R}$. Is $\mathbb{R}^2$ is a vector space over $\mathbb{R}$ under the above operation?*

(1.8) Suggested Readings:*(i) N.S. Gopalakrishnan, University Algebra, New Age International (P) Limited, Publishers.*

(ii) Kenneth Hoffman, Ray Kunze, Linear Algebra,Prentice Hall India.

Lesson-II Subspaces of vector space and quotient space

2.0 **Structure**

(2.1) Introduction: *Given any algebraic structure such as group, ring or a field, we have studied sub-algebraic and quotient structures such a subgroup, a subring or a subfield and quotient group, qoutient ring. Similarly we shall now define subspace of a vector space and quotient vector space.*

(2.2) Objective: *The aim of this lesson is to find new vector spaces, knowing the given vector space.*

(2.3) Subspace of vector space:

(2.3.1) Definition: *A non-empty subset W of a vector space $V(F)$ is said to be a subspace of V if W is itself a vector space under the operations of addition and scalar multiplication defined for V.*

Note *For any vector space V over a field F, the set $\{0\}$ and the set V, both are subsets of V. Also, both of these are vector spaces under the operations of* *addition and scalar multiplication of V. Hence, both $\{0\}$ and V are subspaces of V, known as* **trivial subspaces** *and the subspaces other than $\{0\}$ and V are called* **proper subspaces of** V.

(2.3.2) Theorem: *A non-empty subset W of a vector space V is a subspace of V if and only if (i) $x + y \in W, \ \forall \ x, y \in V$ and (ii) $\alpha x \in W, \forall \ \alpha \ \in \ F$ and $\forall \ x \ \in \ W$.*

proof: *Firstly, we suppose that W is a subspace of V. Then W is itself a vector space under the operations of V. This implies that (i) $x + y \in W, \ \forall \ x, y \in V$ and (ii) $\alpha x \in W, \forall \ \alpha \ \in \ F$ and $\forall \ x \ \in \ W$.*

Conversely, suppose that (i) $x + y \in W, \ \forall \ x, y \in V$ and (ii) $\alpha x \in W, \forall \ \alpha \ \in \ F$ and $\forall \ x \ \in \ W$. We shall prove that W is a subspace of V.

For this, $-1 \in F$ and $x \in W \Rightarrow (-1)x \in W$ (by (ii)) $\Rightarrow -x \in W$. This implies that every element of W has additive inverse.

Now by (i) we have $\forall \ x \ \in W, -x \ \in \ W \Rightarrow x + (-x) \in W \Rightarrow 0 \in W$, so that additive identity exist in W.

Since $W \subset V$, therefore, $x + y = y + x, \forall \ x, \ y \in \ W$

$x + (y + z) = (x + y) + z, \forall \ x, \ y, \ z \in W$

$\alpha(x + y) = \alpha x + \alpha y, \forall \ \alpha \ \in \ F, \ x, \ y \in W$

$(\alpha + \beta)x = \alpha x + \beta x, \forall \ \alpha, \ \beta \in F, \ x \in W$

$\alpha(\beta x) = (\alpha\beta)x, \forall \ \alpha, \ \beta \in F, \ x \in W$

$1x = x, \forall \ x \in W, \ 1 \in F$

$\Rightarrow W$ *is a subspace of V.*

(2.3.3) Theorem: *A non-empty subset W of a vector space V is a subspace of V if and only if (i) $x - y \in W, \ \forall \ x, \ y \in V$ i.e, W is a subgroup of $(V, +)$ and (ii) $\alpha x \in W, \forall \ \alpha \ \in \ F$ and $\forall \ x \ \in \ W$.*

proof: *Firstly, we suppose that* W *is a subspace of* V. *Then* W *is itself a vector space under the operations of* V. *This implies that (i)* $x - y \in W, \ \forall \ x, y \in V$ *and (ii)* $\alpha x \in W, \forall \ \alpha \ \in \ F$ *and* $\forall \ x \ \in \ W$.

Conversely, suppose that (i) $x - y \in W, \ \forall \ x, y \in V$ *and (ii)* $\alpha x \in W, \forall \ \alpha \in F$ *and* $\forall \ x \in W$. *We shall prove that* W *is a subspace of* V.

For this, $-1 \in F$ *and* $x \in W \Rightarrow (-1)x \in W$ *(by (ii))* $\Rightarrow -x \in W$. *This implies that every element of* W *has additive inverse.*

For, $x \in W$ *and* $y \in W \Rightarrow x \in W$ *and* $-y \in W \Rightarrow x - (-y) = x + y \in W$. *So,* W *is closed under* $+$.

Now by (i) we have $\forall \ x \ \in W, \Rightarrow x - x \in W \Rightarrow 0 \in W$, *so that additive identity exist in* W.

Since $W \subset V$, *therefore,* $x + y = y + x, \forall \ x, y \in \ W$

$x + (y + z) = (x + y) + z, \forall \ x, \ y, \ z \in W$

$\alpha(x + y) = \alpha x + \alpha y, \forall \ \alpha \ \in \ F, \ x, \ y \in W$

$(\alpha + \beta)x = \alpha x + \beta x, \forall \ \alpha, \ \beta \in F, \ x \in W$

$\alpha(\beta x) = (\alpha\beta)x, \forall \ \alpha, \ \beta \in F, \ x \in W$

$1x = x, \forall \ x \in W, \ 1 \in F$

$\Rightarrow W$ *is a subspace of* V.

(2.3.4) Theorem: *A non empty subset* W *of* V *is a subspace of* V *if and only if* $\alpha x + \beta y \ \in \ W \ \forall \ \alpha, \ \beta \in F$ *and* $x, \ y \in V$.

Proof: *First, we suppose that* W *is a subspace. Then* W *is itself a vector space. This implies that* $\alpha x + \beta y \ \in \ W \ \forall \ \alpha, \ \beta \in F$ *and* $x, \ y \in V$. *Conversely, suppose that* $\alpha x + \beta y \ \in \ W \ \forall \ \alpha, \ \beta \in F$ *and* $\forall \ x, \ y \in V$*...........(1).*

We shall show that W *is a subspace of* V. *For this, (i) put* $\alpha = 1$ *and* $\beta = -1$ *in (1) we get* $x - y \in W, \forall \ x, y \ \in W$.

Similarly, put $\beta = 0$ *in (1) we get* $\alpha x \in V$, *forall* $\alpha \in F$ *and* $\forall\ x \in V$. *Therefore by the above theorem* W *is a subspace of* V.

(2.3.5) Theorem: *Let* W_1 *and* W_2 *be two subspaces of a vector space* V. *Then* $W_1 \cap W_2$ *is also a subspace of* V *but the* $W_1 \cup W_2$ *need not be a subspace of* V.

Proof. *Since* $0 \in W_1 \cap W_2$, *so* $W_1 \cap W_2 \neq \phi$. *Let* $x, y \in W_1 \cap W_2$ *and* $\alpha, \beta \in F$. *Then* $x, y \in W_1$ *and* $x, y \in W_2$

$\Rightarrow \alpha x + \beta y \in W_1$ *and* $\alpha x + \beta y \in W_2$ *(because* W_1, W_2 *are subspaces).*

$\Rightarrow \alpha x + \beta y \in W_1 \cap W_2$

$\Rightarrow W_1 \cap W_2$ *is a subspace of* V.

Cosider $V = \mathbb{R}^2$ *be a vector space over* $\mathbb{R}$. *Let* $W_1 = \{(x_1, 0)|x_1 \in \mathbb{R}\}$ *and* $W_2 = \{(0, x_2)|x_2 \in \mathbb{R}\}$ *be two subsets of* V. *Then we shall prove that* W_1 *and* W_2 *are subspaces of* V *and* $W_1 \cup W_2$ *is not a subspace of* V.

For this, let $\alpha, \beta \in \mathbb{R}$ *and* $x, y \in W_1$. *Then* $\alpha x + \beta y = \alpha(x_1, 0) + \beta(y_1, 0) = (\alpha x_1, 0) + (\beta y_1, 0) = (\alpha x_1 + \beta y_1, 0) \in W_1$

$\Rightarrow \alpha x + \beta y$.

Similarly, let $\alpha, \beta \in \mathbb{R}$ *and* $x, y \in W_2$.

Then $\alpha x + \beta y = \alpha(0, x_1) + \beta(0, y_1) = (0, \alpha x_1 + \beta y_1) \in W_2$

$\Rightarrow \alpha x + \beta y \in W_2$.

Now $(1, 0) \in W_1 \cup W_2$ *and* $(0, 1) \in W_1 \cup W_2$ *but* $(1, 0) + (0, 1) = (1, 1) \notin W_1 \cup W_2$.

This shows that $W_1 \cup W_2$ *is not a subspace of* V.

(2.3.6) Theorem: *Let* W_1 *and* W_2 *be two subspaces of a vector space* $V(F)$. *Then* $W_1 \cup W_2$ *is a subspace of* V *if and only if either* $W_1 \subset W_2$ *or* $W_2 \subset W_1$.

Proof. *First, let us suppose that either* $W_1 \subset W_2$ *or* $W_2 \subset W_1$. *When* $W_1 \subset W_2$, *then* $W_1 \cup W_2 = W_2$, *which is a subspace of* V. *Similarly if* $W_2 \subset W_1$, *then* $W_1 \cup W_2 = W_1$ *which is a subspace of* V. *This implies that* $W_1 \cup W_2$ *is a subspace of* V.

Conversely, suppose that $W_1 \cup W_2$ *is a subspace of* V. *Then we shall show that either* $W_1 \subset W_2$ *or* $W_2 \subset W_1$.

For this, suppose that $W_1 \not\subset W_2$ *and* $W_2 \not\subset W_1$. *Then there exists* $x \in W_1$ *but* $x \notin W_2$ *and* $y \in W_2$ *but* $y \notin W_1 \Rightarrow x, y \in W_1 \cup W_2$.

Since $W_1 \cup W_2$ *is a subspace of* V, *therefore,* $x - y \in W_1 \cup W_2$

$\Rightarrow$ *either* $x - y \in W_1$ *or* $x - y \in W_2$.

When $x - y \in W_1$, *then* $x - (x - y) \in W_1$ (*because* $x \in W_1$)

$\Rightarrow y \in W_1$, *which is a contradiction to the fact that* $y \notin W_1$. *Similarly, when* $x - y \in W_2 \Rightarrow y + (x - y) \in W_2$ (*because* $y \in W_2$)

$\Rightarrow x \in W_2$, *which is a contradiction to the fact that* $x \notin W_2$. *This implies that our supposition is wrong.*

Hence, either $W_1 \subset W_2$ *or* $W_2 \subset W_1$.

(2.3.7) Theorem: *Let* W_1 *and* W_2 *be two subspaces of a vector space* $V(F)$. *Then* $W_1 + W_2 = \{x_1 + y_1 | x_1 \in W_1, y_1 \in W_2\}$ *is also a subspace of* V.

Proof. *Since* $0 + 0 = 0$ *in* V, *so* $0 \in W_1 + W_2 \Rightarrow W_1 + W_2 \neq \phi$.

Let $u, v \in W_1 + W_2$ *and* $\alpha, \beta \in F$. *Then* $u = x_1 + y_1$ *and* $v = x_2 + y_2$, *where* $x_1, x_2 \in W_1$ *and* $y_1, y_2 \in W_2$.

$$\begin{aligned} \text{Now, } \alpha u + \beta v &= \alpha(x_1 + y_1) + \beta(x_2 + y_2) \\ &= (\alpha x_1 + \beta x_2) + (\alpha y_1 + \beta y_2) \in W_1 + W_2 \end{aligned}$$

$\Rightarrow \alpha u + \beta v \in W_1 + W_2$. *This shows that* $W_1 + W_2$ *is a subspace of* V.

(2.3.8) Theorem: *Let* V *be a vector space over a field* F *and* $\cap_{\alpha \in \Delta} W_\alpha$ *is also a subspace of* V.

Proof. *Since* $W_\alpha \neq \phi$, *for each* $\alpha \in \Delta$. *Therefore,* $W = \cap_{\alpha \in \Delta} W_\alpha \neq \phi$. *Now, let* $x \in W$ *and* $y \in W$.

Then $x,\, y \in W_\alpha$ *for each* $\alpha \in \Delta$. *Since* W_α *is a subspace for each* α, *we have* $x - y \in W_\alpha$ *and* $ax \in W_\alpha$ *for each* $\alpha \in \Delta$.

This implies that $x - y \in W$ *and* $ax \in W$. *Hence* W *is a subspace of* V.

Corollary *Let* S *be any subset of a vector space* V. *Then the intersection of all subspaces of* V *containing* S *is a subspace of* V *containing* S.

Proof *Since* $S \subset W_\alpha$ *for each* $\alpha \in \Delta$. *Therefore* $S \subset \cap_{\alpha\in\Delta} W_\alpha$ *and by theorem* $\cap_{\alpha\in\Delta} W_\alpha$ *is a subspace containing* S.

(2.4) Examples

1. *The intersection of any number of subspaces of a vector space* $V(F)$ *is a subspace of* V.

Solution *Let* $\mathcal{F} = \{W_\alpha | \alpha \in \Delta\}$ *be any family of subspaces of* V. *Then*

$$0 \in \cap_{\alpha\in\Delta} W_\alpha \Rightarrow \cap_{\alpha\in\Delta} W_\alpha \neq \phi.$$

Let $a,\, b \in F$ *and* $x,\, y \in \cap_{\alpha\in\Delta} W_\alpha$. *Then* $x,\, y \in W_\alpha,\, \forall \alpha \in \Delta$

$\Rightarrow ax + by \in W_\alpha,\, \forall \alpha \in \Delta \Rightarrow ax + by \in \cap_{\alpha\in\Delta} W_\alpha$.

Hence $\cap_{\alpha\in\Delta} W_\alpha$ *is a subspace of* V.

2. *Let* $V = \mathbb{R}^n$ *be a vector space over* $\mathbb{R}$. *If* $W = \{x = (x_1,\, \ldots,\, x_n) |\ x_1 = 0\}$, W *is a subspace of* V.

Solution: *Since* $\mathbf{0} = (0,\, 0\, \ldots,\, 0) \in W$. *So,* $W \neq \phi$.

let $x,\, y \in W$ *and* $\alpha,\, \beta \in F$. *Then* $x = (x_1,\, \ldots,\, x_n)$ *and* $y = (y_1,\, \ldots,\, y_n)$ *such that* $x_1 = 0 = y_1$..........(1).

Now $\alpha x + \beta y = (\alpha x_1,\, \ldots,\, \alpha x_n) + (\beta y_1,\, \ldots,\, \beta y_n)$

$\qquad = (\alpha x_1 + \beta y_1,\, \ldots,\, \alpha x_n + \beta y_n)$, *using* (1) *we get*

$\alpha x + \beta y = (\alpha x_1 + \beta y_1,\, \ldots,\, \alpha x_n + \beta y_n)$

$\qquad = (0,\, \alpha x_2 + \beta y_2,\, \ldots,\, \alpha x_n + \beta y_n)$.

This implies that $\alpha x + \beta y \ \in\ W$. *Hence* W *is a subspace of* V.

3. *Let* $V = F[x]$ *be a vector space of polynomials in* x *over* F *and* $W = F_n[x]$ *be the subset of all polynomials of degree less than or equal to* n*. Then* W *is a subspace of* V*.*

Solution: *Since* $0 \in W$ *as the* $\deg 0 = -\infty$*. Therefore,* $W \neq \phi$*.*

Now, let $\alpha,\, \beta \in F$ *and* $f(x),\, g(x) \in V$*. Then* $\deg(f(x)) \leq n$ *and* $\deg(g(x)) \leq n \Rightarrow \deg(\alpha f(x) + \beta g(x)) \leq n$

$\Rightarrow \alpha f(x) + \beta g(x) \in W$*. Hence* W *is a subspace of* V*.*

3. *Let* V *be a vector space of real valued functions over* $\mathbb{R}$*. Show that* $W = \{f(x) \in V | \frac{d^2 f}{dx^2} + a\frac{df}{dx} + bf = 0$ *where* $a,\, b$ *are fixed reals* $\}$*.*

Solution *Since* $O(x) = 0,\, \forall\ x$ *and* $\frac{d^2 O}{dx^2} + a\frac{dO}{dx} + bO = 0$

$\Rightarrow O \in W \Rightarrow W \neq \phi$*. Let* $\alpha,\, \beta \in \mathbb{R}$ *and* $f(x),\, g(x) \in W$

$\Rightarrow \frac{d^2 f}{dx^2} + a\frac{df}{dx} + bf = 0$ *and* $\frac{d^2 g}{dx^2} + a\frac{dg}{dx} + bg = 0$ *.*

Now $\frac{d^2(\alpha f + \beta g)}{dx^2} + a\frac{d(\alpha f + \beta g)}{dx} + b(\alpha f + \beta g) = \frac{d^2 \alpha f}{dx^2} + a\frac{d\alpha f}{dx} + b\alpha f + \frac{d^2 \beta g}{dx^2} + a\frac{d\beta g}{dx} + b\beta g$

$= \frac{\alpha d^2 f}{dx^2} + a\frac{\alpha df}{dx} + \alpha bf + \frac{\beta dg}{dx^2} + a\frac{\beta dg}{dx} + \beta g$

$= \alpha(\frac{d^2 f}{dx^2} + a\frac{df}{dx} + bf) + \beta(\frac{d^2 g}{dx^2} + a\frac{dg}{dx} + bg) = 0 + 0 = 0$

$\Rightarrow \alpha f(x) + \beta g(x) \in W$*. Hence* W *is a subspace of* V*.*

4. *Let* $V = \{A | A = [a_{ij}]_{n\times n},\, a_{ij} \in \mathbb{R}\}$ *be a vector space over* $\mathbb{R}$*. Show that* W*, the set consisting of all the symmetric matrices is a subspace of* V*.*

Solution. *Since* $O = [0]_{n\times n} \in W \Rightarrow W \neq \phi$*. Let* $\alpha,\, \beta \in F$ *and* $P,\, Q \in W$*. Then* $P = [p_{ij}]$ *and* $Q = [q_{ij}]$ *such that* $p_{ij} = p_{ji}$ *and* $q_{ij} = q_{ji}$*.*

Now $\alpha P + \beta Q = [\alpha p_{ij}] + [\beta q_{ij}] = [\alpha p_{ij} + \beta q_{ij}] = [\alpha p_{ji} + \beta q_{ji}] = [r_{ij}]$*, where* $r_{ij} = \alpha p_{ji} + \beta q_{ji} = \alpha p_{ji} + \beta q_{ji} = r_{ji} \Rightarrow [r_{ij}]$ *is a symmetric matrix. Therefore,* $\alpha P + \beta Q \in W$*.*

5. *Let* $a,\, b,\, c$ *be fixed elements of a field* F*. Show that* $W = \{(x,\, y,\, z) | ax + by + cz = 0; x,\, y,\, z \in F\}$ *is a subspace of* F^3*.*

Solution *Since* $(0,\, 0,\, 0) \in W$ *as* $a0 + b0 + c0 = 0,\, 0 \in F$

$\Rightarrow W \neq \phi$.

Let $\alpha, \beta \in F$ *and* $w_1, w_2 \in W$. *Then* $w_1 = (x_1, y_1, z_1)$ *and* $w_2 = (x_2, y_2, z_2)$ *such that* $ax_1 + by_1 + cz_1 = 0$ *and* $ax_2 + by_2 + cz_2 = 0$.

Now $\alpha w_1 + \beta w_2 = \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)$

$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$ *and*

$a(\alpha x_1 + \beta x_2) + b(\alpha y_1 + \beta y_2) + c(\alpha z_1 + \beta z_2) = \alpha(ax_1 + by_1 + cz_1) + \beta(ax_2 + by_2 + cz_2) = \alpha 0 + \beta 0 = 0 + 0 = 0$

$\Rightarrow \alpha w_1 + \beta w_2 \in W$

$\Rightarrow W$ *is a subspace of* V.

6. *Let* $V = \mathbb{R}^3$ *be a vector space over* $\mathbb{R}$. *Which of the following subsets of* V *are subspaces?*

(*i*)$W = \{(x_1, x_2, x_3) | x_1 \leq 0\}$

(*ii*)$W_1 = \{(x_1, x_2, x_3) | x_3$ *is an integer* $\}$

(*iii*)$W_2 = \{(x_1, x_2, x_3) | x_1, x_2, x_3 \in \mathbb{Q}\}$

(*iv*)$W_3 = \{(x_1, x_2, x_3) | x_1 \geq x_2 \geq x_3\}$.

Solution(*i*) *Let* $(x_1, x_2, x_3) \in V$. *Then* $x_1 \leq 0$. *Take* $\alpha = -2$. *Then* $\alpha(x_1, x_2, x_3) = (\alpha x_1, \alpha x_2, \alpha x_3) = (-2x_1, -2x_2, -2x_3) \notin W$ *because* $-2x_1 > 0$. *This shows that* W *is not a subspace of* V.

(*ii*) *Let* $(x_1, x_2, x_3) \in W_1$. *Then* x_3 *is an integer. Now for* $\sqrt{2} \in \mathbb{R}$, $\sqrt{2}(x_1, x_2, x_3) = (\sqrt{2}x_1, \sqrt{2}x_2, \sqrt{2}x_3) \notin W$ *as* $\sqrt{2}x_3$ *is not an integer. Therefore* W_1 *is not a subspace of* V.

(*iii*) *Let* $(x_1, x_2, x_3) \in W_2$. *Then* $x_1, x_2, x_3 \in \mathbb{Q}$. *Now for* $\pi \in \mathbb{R}$, $\pi(x_1, x_2, x_3) = (\pi x_1, \pi x_2, \pi x_3)$, $\pi x_1, \pi x_2, \pi x_3$ *need not be rational numbers. Therefore,* W_2 *is not a subspace of* V.

(*iv*) *Let* $(x_1, x_2, x_3) \in W_3$. *Then* $x_1 \geq x_2 \geq x_3$. *Now for* $\alpha = -1$, *we have* $-1(x_1, x_2, x_3) = (-x_1, -x_2, -x_3) \notin W_3$. *Therefore,* W_3 *is not a subspace of*

V.

(2.5) Let Us Sum Up: *In this lesson we have defined subspace of a vector space and discussed the critera for a non-empty subset of vector space to be subspace. Then we have illustrated subspace with various examples.*

(2.6) Lesson End Exercise

1. *If V_1 is a subspace of V_2 and V_2 is a subspace of V, show that V_1 is a subspace of V.*

2. *Which of the following spaces are subspaces of $\mathbb{R}^n$? Why?*

(i) $W_1 = \{x = (x_1, x_2, \ldots, x_n) | x_1 > x_2\}$.

(ii) $W_2 = \{x = (x_1, x_2, \ldots, x_n) | x_1 + x_2 + \ldots + x_n = 0\}$.

(iii) $W_3 = \{x = (x_1, x_2, \ldots, x_n) | x_1 + x_2 + \ldots + x_n = 1\}$.

(iv) $W_4 = \{x = (x_1, x_2, \ldots, x_n) | x_1 = 0 \text{ and } x_n = 0\}$.

(iv) $W_5 = \{x = (x_1, x_2, \ldots, x_n) | x_1 = 2x_2 + 3x_3 + \ldots + nx_n\}$.

Answer *(ii), (iv) and (v).*

3. *Let $V = C[0, 1]$ be the space of continuous functions on $[0, 1]$. Which of the following are subspaces of V? and why?.*

(i) $W = \{x | x \in V, x(t) \geq 0\}$.

(ii) $W_1 = \{x | x \in V, x(t^2) = x(t)^2\}$

(iii) $W_2 = \{x | x \in V, x(t) = x(-t)\}$

(iv) $W_3 = \{x | x \in V, x(t) \text{ is a polynomial of degree } 3\}$.

Answer *(iii) and (iv).*

4. *Show that $\mathbb{R}^2$ is not a subspace of a vector space $\mathbb{R}^3$. Justify your answer.*

5. *Let W_1 and W_2 be two subspaces of V. Then show that $W_1 + W_2$ is the smallest subspace of V containing $W_1 \cup W_2$.*

(2.7) Quotient Space

Exercise *Let V be a vector space over a field F and W be its subspace. Then show that the set of cosets of W in V , $W' = \{x + W | x \in V\}$ is a vector space under the operation additions $(+)$ and scalar multiplication defined as $(x+W)+(y+W) = (x+y)+W, \forall\ x,\ y \in V$ and $\alpha(x+W) = \alpha x + W, \forall \alpha \in F,\ x \in V$ respectively.*

Solution. Properties under addition $+$:

(i)$+$ **is binary operation**: *Let $x + W = x' + W$ and $y + W = y' + W$. Then $x - x' \in W$ and $y - y' \in W \Rightarrow (x - x') + (y - y') \in W \Rightarrow (x + y) - (x' + y') \in W$ $\Rightarrow (x + y) + W = (x' + y') + W$*

$\Rightarrow (x + W) + (y + W) = (x' + W) + (y' + W)$. This shows that $+$ is a binar operation on W'.

(ii) $(x + W + y + W) + z + W = ((x + y) + W) + (z + W) = ((x + y) + z) + W$ $= (x + (y + z)) + W = x + W + (y + W + z + W), \forall\ x,\ y,\ z \in V$.

(iii) There exists $0 + W \in W'$ such that $(x + W) + (0 + W) = (x + 0) + W = x + W, \forall\ x + W \in W'$.

(iv) For each $x + W \in W'$ there exists $-x + W \in W'$ such that $(x + W) + (-x + W) = (x - x) + W = 0 + W$.

(v) $(x + W) + (y + W) = (x + y) + W$
$= (y + x) + W = (y + W) + (x + W), \forall\ x,\ y \in V$.

Properties under scalar multiplication:

Let $\alpha,\ \beta \in F$ and $x + W,\ y + W \in W'$. Then

(i) $(\alpha + \beta)(x + W) = (\alpha + \beta)x + W$
$= (\alpha x + \beta x) + W = (\alpha x + W) + (\beta x + W) = \alpha(x + W) + \beta(x + W)$.

(ii) $\alpha(x + W + y + W) = \alpha((x + y) + W) = \alpha(x + y) + W$
$= (\alpha x + \alpha y) + W = (\alpha x + W) + (\alpha y + W) = \alpha(x + W) + \beta(y + W)$.

(iii) $(\alpha\beta)(x + W) = (\alpha\beta)x + W = \alpha(\beta x + W) = \alpha(\beta(x + W))$.

(iv) $1(x+W) = x+W$.

Thus, W' is a vector space over F.

Definition (2.7.1) *Let V be a vector space over a field F and W be its subspace. Then the space of cosets of W in V is called a quotient space under the addition and scalar multiplication defined as $(x+W)+(y+W) = (x+y)+W, \forall\ x,\ y \in V$ and $\alpha(x+W) = \alpha x + W, \forall \alpha \in F,\ x \in V$ respectively. It is denoted by V/W.*

(2.8) University Model Questions

1. *Let U and W be subspaces of V such that $W \subset U$, then prove that U/W is subspace of V/W.*

2. *Given a subspace S of a vector space V over F, show how to make the additive subgroup V/S into a vector space over F.*

3. *Show that $W = \{(x_1,\ x_2,\ \ldots,\ x_n)|x_1 = x_2 + x_3 + \ldots + x_n\}$ is a subspace of $\mathbb{R}^n$.*

4. *Show that $W = \{(x_1,\ x_2,\ \ldots,\ x_n)|x_1 \geq 0\}$ is not a subspace of $\mathbb{R}^n$.*

(2.9) Suggested Readings:*(i) N.S. Gopalakrishnan, University Algebra, New Age International (P) Limited, Publishers.*

(ii) Kenneth Hoffman, Ray Kunze, Linear Algebra,Prentice Hall India.

Lesson-III

Linear dependence and linear independence of set of vectors

(3.1) Introduction: *In this lesson we shall study the fundamental properties of vectors or set of vectors i.e whether a set of vectors is linearly independent or linearly dependent.*

(3.2) Objective: *The students will learn the geometric properties of vectors by going through this lesson.*

(3.3) Linear dependence and linear independence of vectors

(3.3.1) Definition: *Let $x_1, x_2, \ldots, x_n$ be elements of a vector space $V(F)$ and $a_1, a_2, \ldots, a_n \in F$. Then $a_1x_1 + a_2x_2 + \ldots + a_nx_n$ is called a linear combination of $x_1, x_2, \ldots, x_n$.*

(3.3.2) Definition: *A subset $S = \{x_1, x_2, \ldots, x_n\}$ of a vector space $V(F)$ is said to be linearly dependent if there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zero such that $\alpha_1x_1 + \alpha_2x_2 + \ldots + \alpha_nx_n = 0$. The vectors $x_1, x_2, \ldots, x_n$ are called linearly dependent vectors.*

(3.3.3) Definition*: A subset $S = \{x_1, x_2, \ldots, x_n\}$ of a vector space $V(F)$ is said to be linearly independent if $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0$, for $\alpha_i \in F, \forall\ i \Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$. The vectors $x_1, x_2, \ldots, x_n$ are called linearly independent vectors.*

Remarks*: (i) Any subset $S \subset V$ which contains a linearly dependent set is linearly dependent.*

For this, let $S_1 = \{x_1, \ldots, x_k\}$ be linearly dependent subset of $S = \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\}$. Then there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ not all zero such that $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k = 0$

$\Rightarrow \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k + 0x_{k+1} + \ldots + 0x_n = 0.$

This shows that S is linearly dependent.

(ii) Any subset S of V which contains 0 is linearly dependent.

In this case we have a linear combination $1(0) + 0x_1 + 0x_2 + \ldots + 0x_n = 0, \forall\ x_i \in S, i = 1, 2, \ldots, n \Rightarrow S$ is linearly dependent.

(iii) Let $0 \neq v \in V$. Then $\{v\}$ is linearly independent.

For this, suppose that $\alpha v = 0$. Then either $\alpha = 0$ or $v = 0$. Since $v \neq 0$, so $\alpha = 0$. Hence $\{v\}$ is linearly independent.

(iv) Every subset of linearly independent set is linearly independent.

For this, Suppose contrary that any subset S_1 of linearly independent set S is not linearly independent. Then S_1 is L.D. subset of $S \Rightarrow$ by remark (i) S is also linearly dependent, which is a contradiction. Hence every subset of a linearly independent set is linearly independent.

(3.3.4) Theorem*: Let V be a vector space over F. Then*

(i) the set $\{x_1, x_2\}$ is linearly dependent if and only if one is a scalar multiple of other.

(ii) the set $\{x_1, x_2, x_3\}$ is linearly dependent if and only if one is a linear

combination of other two.

Proof. (i) First, let's suppose that $\{x_1, x_2\}$ *is linearly dependent.*

Then there exists scalars α, β *(not both zero) such that* $\alpha x_1 + \beta x_2 = 0$.

Suppose that $\alpha \neq 0$,

then there exists $\alpha^{-1} \in F$ *such that* $\alpha^{-1}(\alpha x_1 + \beta x_2) = 0$

$\Rightarrow \alpha^{-1}\alpha x_1 + \alpha^{-1}\beta x_2) = 0 \Rightarrow x_1 + \alpha^{-1}\beta x_2 = 0 \Rightarrow x_1 = -\alpha^{-1}\beta x_2$. *This proves the direct part.*

Conversely, suppose that $x_1 = kx_2 \Rightarrow x_1 - kx_2 = 0$

$\Rightarrow 1(x_1) + (-k)x_2 = 0$, *which is a non-trivial linear relation.*

This shows that $\{x_1, x_2\}$ *is linear dependent.*

(ii) Suppose that the set $\{x_1, x_2, x_3\}$ *is linearly dependent. Then there exists scalars* α, β, γ *(not all zero) such that* $\alpha x_1 + \beta x_2 + \gamma x_3 = 0$.

Without loss of generality, suppose that $\alpha \neq 0$. *Then there exists* $\alpha^{-1} \in F$ *such that* $\alpha^{-1}(\alpha x_1 + \beta x_2 + \gamma x_3) = 0$

$\Rightarrow \alpha^{-1}\alpha x_1 + \alpha^{-1}\beta x_2 + \alpha^{-1}\gamma x_3 = 0$

$\Rightarrow x_1 + \alpha^{-1}\beta x_2 + \alpha^{-1}\gamma x_3 = 0$

$\Rightarrow x_1 = -\alpha^{-1}\beta x_2 - \alpha^{-1}\gamma x_3$.

This proves the direct part.

Conversely, suppose that one vector is a linear combination of other two, say $x_1 = \alpha x_2 + \beta x_3$

$\Rightarrow x_1 - \alpha x_2 - \beta x_3 = 0$, *which is a non trivial linear relation among the vectors.*

This implies that the set $\{x_1, x_2, x_3\}$ *is linearly dependent.*

(3.3.5) Theorem*: Let* V *be a vector space over* F. *Then a subset* $S = \{x_1, x_2, \ldots, x_n\}$ *is linearly dependent if and only if some element of* S *is linear combination of others.*

Proof. *Suppose that the set* $S = \{x_1, x_2, \ldots, x_n\}$ *is linearly dependent. Then* *there exists scalars* $a_1, a_2, \ldots, a_n \in F$ *(not all zero) such that* $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$.

Without loss of generality, assume that $a_n \neq 0$.

Then there exists ${a_n}^{-1} \in F$ *such that*

${a_n}^{-1}(a_1x_1 + a_2x_2 + \ldots + a_nx_n) = 0$

$\Rightarrow {a_n}^{-1}a_1x_1 + {a_n}^{-1}a_2x_2 + \ldots + {a_n}^{-1}a_nx_n = 0$

$\Rightarrow {a_n}^{-1}a_1x_1 + {a_n}^{-1}a_2x_2 + \ldots + x_n = 0$

$\Rightarrow x_n = -{a_n}^{-1}a_1x_1 - {a_n}^{-1}a_2x_2 - \cdots - {a_n}^{-1}a_{n-1}x_{n-1}$

$\Rightarrow x_n = (-{a_n}^{-1}a_1)x_1 + (-{a_n}^{-1}a_2)x_2 + \cdots + (-{a_n}^{-1}a_{n-1})x_{n-1}$.

This shows that one vector is a linear combination of others.

Conversely, suppose that one vector is a linear combination of others say

$x_1 = a_2x_2 + a_3x_3 + \cdots + a_nx_n$

$\Rightarrow x_1 - a_2x_2 - a_3x_3 - \ldots - a_nx_n = 0$

$\Rightarrow x_1 + (-a_2)x_2 + (-a_3)x_3 + \ldots + (-a_n)x_n = 0$, *which is a non trivial linear relation among the vectors. This implies that the set* $\{x_1, x_2, \ldots, x_n\}$ *is linearly dependent.*

(3.3.6) Theorem*: Let* V *be a vector space over* F*. Then a subset* $S = \{x_1, x_2, \ldots, x_n\}$ *of non-zero vectors is linearly dependent if and only if some vector* x_m, $2 \leq m \leq n$ *can be expressed as a linear combination of its preceeding vectors.*

Proof. *Suppose that* $S = \{x_1, x_2, \ldots, x_n\}$ *is linearly dependent. Then there scalars* $a_1, a_2, \ldots, a_n \in F$ *(not all zero) such that* $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$*....(1)*

Let m *be the largest suffix of* a *for which* $a_m \neq 0$.

Then equation (1) can be written as $a_1x_1+a_2x_2+\ldots+a_mx_m+0x_{m+1}+\ldots 0x_n = 0$

$\Rightarrow a_1x_1 + a_2x_2 + \ldots + a_mx_m = 0$

. If $m = 1$, then $a_1x_m = 0 \Rightarrow x_m = 0$ which is contradiction to the fact that all vectors of S are non zero.

Therefore, $m > 1$ or $2 \leq m \leq n$. Now, there exists ${a_m}^{-1} \in F$ such that ${a_m}^{-1}(a_1x_1 + a_2x_2 + \ldots + a_mx_m) = 0$

$\Rightarrow ({a_m}^{-1}a_1)x_1 + ({a_m}^{-1}a_2)x_2 + \ldots + ({a_m}^{-1}a_m)x_m = 0$

$\Rightarrow ({a_m}^{-1}a_1)x_1 + ({a_m}^{-1}a_2)x_2 + \ldots + x_m = 0$

$\Rightarrow x_m = (-{a_m}^{-1}a_1)x_1 + (-{a_m}^{-1}a_2)x_2 + \ldots + (-{a_m}^{-1}a_{m-1})x_{m-1}$.

This shows that x_m is a linear combination of its preceeding vectors. Conversely, suppose that some vector x_m can be written as linear combination of its preceeding vectors. Then $x_m = a_1x_1 + a_2x_2 + \ldots + a_{m-1}x_{m-1}$, for $a_1, a_2, \ldots, a_m \in F$

$\Rightarrow a_1x_1 + a_2x_2 + \ldots + a_{m-1}x_{m-1} + (-1)x_m + 0x_{m+1} + \ldots + 0x_m = 0$

$\Rightarrow$ *there exists scalars $a_1, a_2, \ldots, a_m = -1 \neq 0, a_{m+1} = \ldots = a_n = 0$ not all zero such that $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$. Hence S is linearly dependent.*

(3.4) Examples

1. *If $V = \mathbb{R}^3$, then*

$$S = \{(1, 1, 0), (0, -1, 1), (-1, 0, -1)\}$$

is linearly dependent because $(1, 1, 0) + (0, -1, 1) + (-1, 0, -1) = (1 - 1, 1 - 1, 1 - 1) = (0, 0, 0)$.

2. *If $V = \mathbb{R}^3$, then*

$$S = \{(1, 1, 0), (0, -1, -1), (-1, 0, -1)\}$$

is linearly independent (L.I.).

Solution *consider* $a(1, 1, 0) + b(0, -1, -1) + c(-1, 0, -1) = (0, 0, 0)$

$\Rightarrow (a - c, a - b, -b - c) = (0, 0, 0)$

$\Rightarrow a - c = 0,\ a - b = 0\ \textit{and}\ -b - c = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, $\begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & -1 \end{vmatrix} = 2 \neq 0$

$\Rightarrow$ *therefore the equations have only trivial solution i.e* $a = 0,\ b = 0,\ c = 0.$

This shows that S is L.I.

3. *Prove that the set*

$$S = \{(1,\ 2,\ 3),\ (1,\ -3,\ 2),\ (2,\ -1,\ 5)\}$$

is linearly dependent in $V = \mathbb{R}^3$.

Solution. *Cosider* $a(1,\ 2,\ 3) + b(1,\ -3,\ 2) + c(2,\ -1,\ 5) = (0,\ 0,\ 0)$ *for* $a,\ b,\ c \in \mathbb{R}$

$\Rightarrow (a + b + 2c,\ 2a - 3b - c,\ 3a + 2b + 5c) = (0,\ 0,\ 0)$

$\Rightarrow a + b + 2c = 0,\ 2a - 3b - c = 0\ \textit{and}\ 3a + 2b + 5c = 0$

these equations can be written in the matrix form as

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, $\begin{vmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ 3 & 2 & 5 \end{vmatrix} = 1(-15 + 2) - 1(10 + 3) + 2(4 + 9) = 0$

$\Rightarrow$ *this system of equations have non-trivial solutions.*

$\Rightarrow$ *We get the scalars* $a,\ b,\ c \in \mathbb{R}$ *not all zero such that* $a(1,\ 2,\ 3) + b(1,\ -3,\ 2) + c(2,\ -1,\ 5) = (0,\ 0,\ 0)$.

Therefore, S is linearly dependent.

4. *If* x *is a linear combination of* $x_1, x_2, \ldots, x_n$ *then show that* $x_1, x_2, \ldots, x_n, x$ *are linearly dependent.*

Solution. *We have* $x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$

$\Rightarrow \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n - 1x = 0$ *which is a non trivial linear relation. This shows that* $x_1, x_2, \ldots, x_n$ *are linearly dependent.*

5. *Let* x, y, z *be linearly independent vectors in a vector space* $V(F)$*. Then* $x+y, y+z, x+z$ *are also linearly independent.*

Solution. *Consider* $\alpha(x+y) + \beta(y+z) + \gamma(x+z) = 0$

$\Rightarrow (\alpha+\gamma)x + (\alpha+\beta)y + (\beta+\gamma)z = 0$

$\Rightarrow \alpha+\gamma = 0, \beta+\gamma = 0$ *and* $\alpha+\beta = 0.$

Solving these equations, we get $\alpha+\beta = 0$ *and* $\alpha-\beta = 0$

$\Rightarrow$ *we get* $\alpha = 0, \beta = 0$ *and* $\gamma = 0$*. Hence* $x+y, y+z, x+z$ *are linearly independent.*

6. *Find the condition under which the vectors* $(b, 1, 0), (1, b, 1), (0, 1, b)$ *are linearly dependent in* $\mathbb{R}^3$.

Solution. *Let* $\alpha, \beta, \gamma \in \mathbb{R}$ *such that* $\alpha(b, 1, 0) + \beta(1, b, 1) + \gamma(0, 1, b) = 0$

$\Rightarrow (\alpha b + \beta, \alpha + b\beta, \beta + b\gamma) = (0, 0, 0)$

$\Rightarrow \alpha b + \beta = 0, \alpha + b\beta = 0$ *and* $\beta + b\gamma = 0.$

$$\Rightarrow \begin{bmatrix} b & 1 & 0 \\ 1 & b & 1 \\ 0 & 1 & b \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \ldots.(*)$$

$$Now, \begin{vmatrix} b & 1 & 0 \\ 1 & b & 1 \\ 0 & 1 & b \end{vmatrix} = b(b^2-1) - b = b^3 - 2b$$

The vectors $(b, 1, 0), (1, b, 1), (0, 1, b)$ *are linearly dependent in* $\mathbb{R}^3$ *if the the above system of equations* $(*)$ *have non-trivial solutions.*

That is the system of equations $(*)$ *have non trivial solutions if* $\begin{vmatrix} b & 1 & 0 \\ 1 & b & 1 \\ 0 & 1 & b \end{vmatrix} = 0$

$\Rightarrow b^3 - 2b = 0$, *which is the required condition.*

(3.5) Let Us Sum Up *: The main property of elements of vector space is their linear dependence and linear independence which have been defined and illustrated with the help of examples in this lesson. With the help of theorems, more properties of vectors have been explored.*

(3.6) **Lesson End Exercise**

1. *Determine whether the following subsets of vector space* $V = \mathbb{R}^3$ *are linearly independent:*

(i) $S_1 = \{(1,\, 0,\, 1),\, (1,\, 1,\, 1),\, (0,\, 0,\, 1)\}$.

(ii) $S_2 = \{(2,\, -1,\, 3),\, (4,\, 1,\, -1),\, (2,\, 3,\, -3)\}$.

(iii) $S_3 = \{(1,\, 1,\, 2),\, (-3,\, 1,\, 0),\, (1,\, -1,\, 1),\, (1,\, 2,\, -3)\}$.

(iv) $S_4 = \{(0,\, 1,\, -2),\, (1,\, -1,\, 1),\, (1,\, 2,\, 1)\}$.

2. *Find the condition under which* $z_1 = a + \iota b$, $z_2 = c + \iota d$ *are L.I. over* $\mathbb{C}$.

3. *Let* $S = \{(2,\, -1,\, 0),\, (1,\, 2,\, 1),\, (0,\, 2,\, -1)\}$. *Show that* S *is linearly independent. Express* $(3,\, 2,\, 1)$ *as a linear combination of elements of* S.

4. *Find* k *if the vectors* $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix}$ *are linearly dependent.*

5. *Let* $V = F_4[x]$ *be a vector space of polynomials of degree less than or equal to 4. Then show that the set of polynomials* $\{1+x,\, x+x^2,\, x^2+x^3,\, x^3+x^4,\, x^4-1\}$ *are L.D.*

(3.7) University Model Questions

1. *Let* V *be the vector space of all twice differentaiable functions on* $[0, 1]$. *Find all* $x \in V$, *such that* $x(t)$ *and* $x'(t)$ *are linearly dependent.*

Hint *Since* $x(t)$ *and* $x'(t)$ *are linearly dependent, so one can be written as linear combination of other. Suppose that* $x'(t) = \alpha x(t)$.

Then $\frac{dx(t)}{dt} = \alpha x(t) \Rightarrow \frac{dx}{x} = \alpha dt$.

Now, integrating on bothside, we get

$\log x = \alpha t + c$, *c is constant of integration.*

2. *If* x, y, z *are linearly independent vectors of* V, *then show that* $x + y$, $y + z$, $x + z$ *are also linearly independent.*

(3.8) Suggested text books *:(i) N.S. Gopalakrishnan, University Algebra, New Age International (P) Limited, Publishers.*

(ii) Kenneth Hoffman, Ray Kunze, Linear Algebra,Prentice Hall India.

(4.1) Introduction : *In this lesson, we are introducing a notion of generating set for a vector space and its properties. Basically the idea is to compute the minimal generating set for a vector space and it turns out to be unique.*

(4.2) Objective : *The students will understand the nature of the smallest vector space containg a non-empty set which turns out to be the linear span of that non empty set.*

(4.3) Linear span of vectors

(4.3.1) Definition *Let S be a non-empty subset of a vector space $V(F)$. Then the set of all linear combinations of any finite number of elements of S is called the* **linear span of** S. *It is denoted by $L(S)$ or $< S >$ so that*

$$L(S) = \left\{ \sum_{i=1}^{n} a_i x_i | a_i \in F \text{ and } x_i \in S, \, 1 \leq i \leq n \right\}.$$

Note *If $S = \phi$. Then $L(S) = \{0\}$.*

(4.3.2) Definition *A subset S of a vector space $V(F)$ is said to be a* **generating set** *of the vector space V if $L(S) = V$.*

(4.3.3) Theorem *Let S be a subset of a vector space $V(F)$. Then $L(S)$ is the smallest subspace of V containing S.*

Proof. *Since* $0 = a_1 0 + \ldots + a_n 0$ *for each* $a_i \in F$.

Therefore $0 \in L(S) \Rightarrow L(S) \neq \phi$.

Now let $x,\, y \in L(S)$ *and* $\alpha,\, \beta \in F$.

Then

$$x = \sum_{i=1}^{n} a_i x_i$$

and

$$y = \sum_{i=1}^{m} b_j y_j$$

for all $a_i,\, b_j \in F$ *and* $x_i,\, y_j \in S$.

Now, $\alpha x + \beta y = \alpha(\sum_{i=1}^{n} a_i x_i) + \beta(\sum_{j=1}^{m} b_j y_j)$

$= \sum_{i=1}^{n} \alpha a_i x_i + \sum_{j=1}^{m} \beta b_j y_j$

$= (\alpha a_1)x_1 + \ldots + (\alpha a_n)x_n + (\beta b_1)y_1 + \ldots + (\beta b_m)y_m$

which is a linear combination of finitely many elements of S. This implies that $\alpha x + \beta y \in L(S)$. Therefore $L(S)$ is a subspace of V.

Now, suppose that W is any subspace of V containing S. Then $s \in W$, $\forall\, s \in S$. Since W is a subspace of V, so

$$\sum_{i=1}^{n} a_i s_i \in W,\ \forall\ a_i \in F,\ s_i \in S.$$

This implies that $L(S) \subset W$. Hence $L(S)$ is the smallest subspace of V containing S.

Corollary. *Let $S = \{x_1,\, x_2,\, \ldots,\, x_n\}$ be a finite set.*

Then $L(S) = \{x | \sum_{i=1}^{n} a_i x_i | a_i \in F$ *and* $x_i \in S\}$.

(4.3.4) Theorem *Let S and T be any subsets of a vector space $V(F)$. Then prove that*

(i) $S \subset L(T) \Rightarrow L(S) \subset L(T)$

(ii) $S \subset T \Rightarrow L(S) \subset L(T)$

(iii) S *is a subspace of* $V \Leftrightarrow L(S) = S$

(iv) $L(L(S)) = L(S)$.

Proof. *(i) Let* $x \in L(S)$. *Then* $x = \sum_{i=1}^{n} a_i x_i, \forall\ a_i \in F$ *and* $x_i \in S$ *Since* $S \subset L(T) \Rightarrow x_i \in L(T), \forall x_i \in S$

$\Rightarrow \sum_{i=1}^{n} a_i x_i \in L(T)$ *as* $L(T)$ *is a subspace of* V

$\Rightarrow L(S) \subset L(T)$.

(ii) Let $x \in L(S)$. *Then* $x = \sum_{i=1}^{n} a_i x_i, \forall\ a_i \in F$ *and* $x_i \in S$. *Since* $S \subset T \Rightarrow x = \sum_{i=1}^{n} a_i x_i, \forall\ a_i \in F$ *and* $x_i \in T$

$\Rightarrow x \in L(T)$. *Hence* $L(S) \subset L(T)$.

(iii) Let S *be a subspace of* $V(F)$. *Then we have to prove that* $L(S) = S$. *Since* $L(S)$ *is the smallest subspace of* V *containing* S. *Therefore* $S \subset L(S)...(1)$

Now, let $x \in L(S)$. *Then there exists* $x_1, x_2, \ldots, x_n \in S$ *and* $a_1, a_2, \ldots, a_n \in F$ *such that*

$$x = \sum_{i=1}^{n} a_i x_i, \forall\ a_i \in F \text{ and } x_i \in S.$$

Since S *is a subspace of* V, *so* $x \in S$.

$L(S) \subset S...(2)$

Therefore, from (1) *and* (2), *we have* $L(S) = S$.

(iv) Since $L(S)$ *is a subspace of* V, *so by (iii) we have*

$$L(L(S)) = L(S).$$

(4.4) Examples

1. *Let* S *and* T *be any subsets of a vector space* $V(F)$. *Then*

$$L(S \cup T) = L(S) + L(T).$$

Solution. *Let* $x \in L(S \cup T)$. *Then*

$$\begin{aligned} x &= \sum_{i=1}^{n} a_i x_i, \ \forall\ x_i \in S \cup T \ \textit{and}\ a_i \in F; i = 1,\, 2\, \ldots,\, n \\ &= \sum a_j x_j + \sum a_k x_k \ ; \ x_j \in S \ \textit{and}\ x_k \in T \end{aligned}$$

$\Rightarrow x \in L(S) + L(T)$

$\Rightarrow L(S \cup T) \subset L(S) + L(T).....(i)$.

Conversely, suppose that $y \in L(S) + L(T)$. *Then*

$$\begin{aligned} y &= \sum_{i=1}^{k} a_i y_i + \sum_{j=k+1}^{m} b_j y_j \ ,\forall\ y_i \in S,\, y_j \in T \\ &= \sum_{i=1}^{m} a_i y_i \ \forall\ y_i \in S \cup T \end{aligned}$$

$\Rightarrow y \in L(S \cup T)$

$\Rightarrow L(S) + L(T) \subset L(S \cup T)....(ii)$.

Hence, from (i) *and* (ii) *we have* $L(S \cup T) = L(S) + L(T)$.

2. *Let* $V = \mathbb{R}^3$. *Show that*

$$(1,\, 7,\, -4) \in L((1,\, -3,\, 2),\, (2,\, -1,\, 1)).$$

Solution. *Let* $(1,\, 7,\, -4) = \alpha(1,\, -3,\, 2) + \beta(2,\, -1,\, 1)$. *Then*

$$\begin{aligned} (1,\, 7,\, -4) &= (\alpha,\, -3\alpha,\, 2\alpha) + (2\beta,\, -\beta,\, \beta) \\ &= (\alpha + 2\beta,\, -3\alpha - \beta,\, 2\alpha + \beta) \\ \Rightarrow 1 &= \alpha + 2\beta \\ 7 &= -3\alpha - \beta \\ -4 &= 2\alpha + \beta \end{aligned}$$

Solving these equations, we get $\alpha = -3$ *and* $\beta = 2$. *Therefore* $(1,\, 7,\, -4) = -3(1,\, -3,\, 2) + 2(2,\, -1,\, 1)$

$\Rightarrow (1,\ 7,\ -4) \in L((1,\ -3,\ 2),\ (2,\ -1,\ 1))$.

3. *Find the conditions on* a, b, c *such that* $\begin{bmatrix} a & b \\ -b & c \end{bmatrix}$ *is a linear combination of the matrices* $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ *and* $C = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

Solution. *Let* $\begin{bmatrix} a & b \\ -b & c \end{bmatrix} = \alpha A + \beta B + \gamma C$.

Then

$$\begin{aligned}
\begin{bmatrix} a & b \\ -b & c \end{bmatrix} &= \alpha \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \beta \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \\
\begin{bmatrix} a & b \\ -b & c \end{bmatrix} &= \begin{bmatrix} \alpha & \alpha \\ 0 & -\alpha \end{bmatrix} + \begin{bmatrix} \beta & \beta \\ -\beta & 0 \end{bmatrix} + \begin{bmatrix} \gamma & -\gamma \\ 0 & 0 \end{bmatrix} \\
\begin{bmatrix} a & b \\ -b & c \end{bmatrix} &= \begin{bmatrix} \alpha+\beta+\gamma & \alpha+\beta-\gamma \\ -\beta & -\alpha \end{bmatrix} \\
\Rightarrow a &= \alpha+\beta+\gamma \\
b &= \alpha+\beta-\gamma \\
-b &= -\beta \\
c &= -\alpha
\end{aligned}$$

Solving these equations, we get $\alpha = -c$, $\beta = b$, $\gamma = -c \Rightarrow -c + b - c = a \Rightarrow a - b + 2c = 0$, *which is the required condition.*

4. *Find the value of* k *so that* $(1,\ -2,\ k)$ *becomes a linear combination of vectors* $(3,\ 0,\ -2)$ *and* $(2,\ -1,\ -5)$.

Solution. Let $(1, -2, k) = \alpha(3, 0, -2) + \beta(2, -1, -5)$. *Then*

$$\begin{aligned}
(1, -2, k) &= \alpha(3, 0, -2) + \beta(2, -1, -5) \\
(1, -2, k) &= (3\alpha, 0, -2\alpha) + (2\beta, -\beta, -5\beta) \\
(1, -2, k) &= (3\alpha + 2\beta, -\beta, -2\alpha - 5\beta) \\
\Rightarrow 3\alpha + 2\beta &= 1 \\
-\beta &= -2 \\
-2\alpha - 5\beta &= k
\end{aligned}$$

Solving these equations we get $\beta = 2$, $\alpha = -1 \Rightarrow 2 + (-5)2 = k \Rightarrow k = -8$.

5. *Does* $(1, -3, 5)$ *belong to the linear span of* $S = \{(1, 2, 1), (1, 1, 1, -1), (4, 5, -2)\}$?

Solution. *Suppose that* $(1, -3, 5) \in L(S)$. *Then* $(1, -3, 5) = a(1, 2, 1) + b(1, 1, -1) + c(4, 5, -2)$

$\Rightarrow$

$$\begin{aligned}
(1, -3, 5) &= (a, 2a, a) + (b, b, -b) + (4c, 5c, -2c) \\
(1, -3, 5) &= (a + b + 4c, 2a + b + 5c, a - b - 2c) \\
\Rightarrow a + b + 4c &= 1....(i) \\
2a + b + 5c &= -3....(ii) \\
a - b - 2c &= 5.....(iii)
\end{aligned}$$

Solving these equations (i), (ii) *and* (iii), *we get* $a + c = 3.....(iv)$ *and* $a + c = \frac{2}{3}....(v)$. *From equations* (iv) *and* (v) *it is clear that we could not find* a, b, c *such that* $(1, -3, 5) = a(1, 2, 1) + b(1, 1, -1) + c(4, 5, -2)$.

6. *Let* $V = \mathbb{R}[x]$ *be vector space of polynomials over* $\mathbb{R}$ *and* $S = \{x^2 - 2x + 5, x + 3, 2x^2 - 3x\}$. *Show that* $f(x) = x^2 + 4x - 3$ *is an element of* $L(S)$.

Solution *Let* $f(x) = a(x^2 - 2x + 5) + b(x + 3) + c(2x^2 - 3x)$. *Then*

$$\begin{aligned} x^2 + 4x - 3 &= a(x^2 - 2x + 5) + b(x + 3) + c(2x^2 - 3x) \\ &= (a + 2c)x^2 + (-2a + b - 3c)x + (5a + 3b) \\ \Rightarrow a + 2c &= 1...(1) \\ -2a + b - 3c &= 4...(2) \\ 5a + 3b &= -3...(3) \end{aligned}$$

Solving equations (1) *and* (2), *we get* $-a + 2b = 11....(4)$. *From equations* (3) *and* (4), *we get* $b = 4 \Rightarrow a = -3$ *and* $c = 2$.

Therefore $f(x) = -3(x^2 - 2x + 5) + 4(x + 3) + 2(2x^2 - 3x)$.

7. *Let* $V = \mathbb{C}$ *be a vector space over* R. *Then show that* $S = \{1, \iota\}$ *is a generating set for* V.

proof. *Since* $L(S) \subset V.........(1)$.

Now let $z \in V$. *Then* $z = a + \iota b$, *for* $, a, b \in \mathbb{R}$

$\Rightarrow z = a.1 + b\iota \Rightarrow z \in L(S)....(2)$.

Therefore, from (1) *and* (2), *we get*

$$V = L(S).$$

This implies that S *is a generating set for* V.

8. *Let* $S = \{(1, 1, 0), (0, 2, 0)\}$. *Show that* $W = \{(x_1, x_2, 0) | x_1, x_2 \in \mathbb{R}\}$ *is the subspace of* $\mathbb{R}^3$ *generated by* S.

Solution. *Let* $x, y \in W$ *and* $a, b \in F$. *Then* $ax + by = a(x_1, x_2, 0) + b(y_1, y_2, 0) = (ax_1 + by_1, ax_2 + by_2, 0) \in W$ *as* $ax_1 + by_1, ax_2 + by_2 \in \mathbb{R}$.

Therefore, W *is a subspace of* V.

Now we shall show that $W = L(S)$. *For this, note that* $S \subset W$

$\Rightarrow L(S) \subset W$.

Let $x \in W$. *Then* $x = (x_1, x_2, 0)$.

Let $x = \alpha(1, 1, 0) + \beta(0, 2, 0)$, *for* $\alpha, \beta \in F$. *Then*

$$\begin{aligned} (x_1, x_2, 0) &= (\alpha, \alpha + 2\beta, 0) \\ \Rightarrow x_1 &= \alpha \\ x_2 &= \alpha + 2\beta \\ \Rightarrow \beta &= \frac{x_2 - x_1}{2} \end{aligned}$$

This implies $x = x_1(1, 1, 0) + \frac{x_2 - x_1}{2}(0, 2, 0)$

$\Rightarrow x \in L(S)$. *Hence* $W = L(S)$.

9. *What is the subspace generated by* $(1, 0, 0)$ *and* $(0, 2, 0)$ *in* $\mathbb{R}^3$*?*

Solution. *Let* $S = \{(1, 0, 0), (0, 2, 0)\}$. *Then* $L(S)$ *is the subspace generated by* $(1, 0, 0)$ *and* $(0, 2, 0)$.

Therefore the required subspace is

$$L(S) = \{(x, y, z)|(x, y, z) = \alpha(1, 0, 0) + \beta(0, 2, 0) = (\alpha, 2\beta, 0)\}.$$

This implies that $x = \alpha$, $\beta = \frac{y}{2}$ *and* $z = 0$.

Thus the subspace generated by $(1, 0, 0)$ *and* $(0, 2, 0)$ *is given by*

$$\{(x, y, z)|x(1, 0, 0) + \frac{y}{2}(0, 2, 0)\}.$$

10. *Let* V *be a vector space over a field* F *and* S *is a subset of a vector space* $V(F)$ *such that* $0 \in S$. *Show that* $L(S) = L(S - \{0\})$.

Solution.**Case-I** *If* $S = \{0\}$, *then* $L(S) = \{0\}$ *and* $S - \{0\} = \phi$

$\Rightarrow L(S - \{0\}) = L(\phi) = \{0\}$.

Hence $L(S) = L(S - \{0\})$.

Case-II *When* $S \neq \{0\}$. *Since* $S - \{0\} \subset S$

$\Rightarrow L(S-\{0\}) \subset L(S)$........(1).

Now, let $x \in L(S)$. *If* $x = 0$, *then* $x \in L(S-\{0\})$ *and* $L(S) \subset L(S-\{0\})$ *and we are done.*

If $x \neq 0$, *then*

$$x = \sum_{i=1}^{k} a_i x_i, \quad \textit{where } a_i \in F,\, x_i \in S$$

for $1 \leq i \leq n \Rightarrow$.

$$x = \sum_{i=1}^{k} a_i x_i + \sum_{i=k+1}^{n} a_i 0,$$

for $x_i \in S$, $1 \leq i \leq k$

$$\Rightarrow x = \sum_{i=1}^{k} a_i x_i \Rightarrow x \in L(S-\{0\}).....(2).$$

Therefore from (1) *and* (2), *we have*

$$L(S) = L(S-\{0\}).$$

11. *Let* V *be a vector space over field* $\mathbb{R}$ *and* $x_1,\, x_2 \in V$. *Then*

$$L(\{x_1,\, x_2\}) = L(\{x_1 - x_2,\, x_1 + x_2\}).$$

Proof. *We have*

$$\begin{aligned} L(\{x_1 - x_2,\, x_1 + x_2\}) &= \{a(x_1 - x_2) + b(x_1 + x_2) | a,\, b \in \mathbb{R}\} \\ &= \{(a+b)x_1 + (b-a)x_2\} \\ &= \{\alpha x_1 + \beta x_2 | \alpha = a + b \textit{ and } \beta = b - a\} \\ &= \{\alpha x_1 + \beta x_2 | \alpha,\, \beta \in \mathbb{R}\} \\ &= L(\{x_1,\, x_2\}. \end{aligned}$$

(4.5) Let Us Sum Up: *In this lesson, we have defined span of subset of a vector space $V(F)$ which turns out to be the smallest subspace of V and illustrated with various examples. The linear span of a subset of a vector space is also called as generating set for that subspace.*

(4.6) Lesson End Exercise

1. *Show that* $S = \{(1,\, 2,\, 3),\, (0,\, 1,\, 2),\, (0,\, 0,\, 1)\}$ *spans* $V = \mathbb{R}^3$.

2. *Find the condition on* $a,\, b,\, c$ *such that*

$$(a,\, b,\, c) \in L(\{(1,\, 2,\, 3),\, (-1,\, 2,\, 4)\}).$$

3. *Express the polynomial* $f(x) = x^3 - 3x^2 + x - 7$ *over* $\mathbb{R}$ *as a linear combination of the polynomials* $x^3 - 3x^2 + 1,\, 2x^3 - 2x + 5,\, x - 8$.

4. *Write the vector* $x = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ *as a linear combination of the vectors* $x_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ *and* $x_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

5. *Which of the following polynomials are in* $< x^3,\, x^2 + 2x,\, x^2 + 2,\, -x + 1 >$ *(i)* $3x^2 + x + 5$ *(ii)* $x^3 + 3x^2 + 3x + 7$.

Hint *Let* $3x^2 + x + 5 = a(x^3) + b(x^2 + 2x) + c(x^2 + 2) + d(-x + 1)$. *Then* $3x^2 + x + 5 = a(x^3) + (b + c)x^2 + (2b - d)x + (2c + d)$

$\Rightarrow a = 0$

$b + c = 3$

$2b - d = 1$

$2c + d = 5$

Solving these equations, we get $b + c = 3$ *and* $b + c = 3$.

Take $a = 0,\ b = 1,\ c = 2,\ d = 1.$

We get $3x^2 + x + 5 = 0(x^3) + 1(x^2 + 2x) + 2(x^2 + 2) + 1(-x + 1)$

$\Rightarrow 3x^2 + x + 5 \in < x^3,\ x^2 + 2x,\ x^2 + 2,\ -x + 1 >.$

(4.7) University Model Questions

1. *Let* S *and* S' *be subsets of vector space* V. *Then show that*

$(i) S \subset S' \Rightarrow L(S) \subset L(S') (ii) L(S \cup S') = L(S) + L(S').$

2. *Let* V_1 *and* V_2 *be subspaces of* V. *Then show that* $V_1 + V_2$ *is the subspace generated by* $V_1 \cup V_2$.

(4.8) Suggested text books *:(i) N.S. Gopalakrishnan, University Algebra, New Age International (P) Limited, Publishers.*

(ii) Kenneth Hoffman, Ray Kunze, Linear Algebra,Prentice Hall India.

Unit-II

Lesson-V **Basis and Dimension**

(5.1) **Introduction**: *In this lesson, we study the computation of invariant of vector space $V(F)$ such as number of elements in the minimal generating set for V. Here the minimal means the smallest set under taking subsets.*

(5.2) **Objectives**:*(i) the students shall come to know the fundamental unit of a vector space.*

(ii) knowing this unit of vector space students will know the full vector space, this unit is known as a **basis**.

(5.3) **Basis and Dimension**

(5.3.1) Definition: *A subset B of a vector space $V(F)$ is said to be a* **basis** *of V if (i) $L(B) = V$ and (ii) B is linearly independent.*

(5.3.2) Definition: *The* **dimension** *of a vector space over a field F is defined by the number of elements in a basis of V. It is denoted by $\dim_F(V)$ or $\dim V$.*

(5.3.3) Theorem: *A subset $B = \{x_1, x_2, \ldots, x_n\}$ of a vector space $V(F)$ is a basis if and only if every $x \in V$ can be uniquely expressed as $x = a_1x_1 + a_2x_2 + \ldots + a_nx_n$, $a_i \in F$, $1 \leq i \leq n$.*

Proof *Suppose that $B = \{x_1, x_2, \ldots, x_n\}$ is a basis of V. Let $x \in V$ has two representations as $x = a_1x_1 + a_2x_2 + \ldots + a_nx_n = b_1x_1 + b_2x_2 + \ldots + b_nx_n$. Then*

$$(a_1 - b_1)x_1 + (a_2 - b_2)x_2 + \ldots + (a_n - b_n)x_n = 0$$

$$\Rightarrow a_1 - b_1 = a_2 - b_2 = \ldots = a_n - b_n = 0$$

(because B is linearly independent)

$$\Rightarrow a_1 = b_1,\ a_2 = b_2,\ \ldots,\ a_n = b_n.$$

Hence, every $x \in V$ has the unique linear combination of elements of B.

Conversely, suppose that every $x \in V$ can be uniquely expressed as $x = a_1x_1 + a_2x_2 + \ldots + a_nx_n$, $a_i \in F$, $1 \leq i \leq n$. Then it follows that

$$V = L(B).$$

Now, we shall prove that B is linearly independent. For this, consider $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$.....(1). Then $0 = 0x_1 + 0x_2 + \ldots + 0x_n$......(2). Since $0 \in V$ has the unique linear combination of elements of B.

Therefore, from (1) and (2), we have $a_1 = 0 = \ldots = a_n$. This implies that B is linearly independent and hence a basis for V.

Theorem (5.3.4): *Let V be a vector space of dimension n. Then any $n + 1$ vectors of V are linearly dependent.*

Proof *We shall prove this theorem by induction on n. When $n = 1$, then $\dim V = 1$. Suppose that $B_1 = \{e_1\}$ is a basis of V and x_1, x_2 be any two elements of V. Then $x_1 = \alpha e_1$ and $x_2 = \beta e_1$.*

Case I *If* $x_1 = 0$ *or* $x_2 = 0$, *then* $1x_1 + 0x_2 = 0$ *or* $0x_1 + 1x_2 = 0$ *is a non-trivial relation between* x_1 *and* x_2. *This implies that* x_1, x_2 *are linearly dependent. Hence the theorem is true for* $n = 1$.

Case II *If* $x_1 \neq 0$ *and* $x_2 \neq 0$. *Then* $x_1 = \alpha e_1$ *and* $x_2 = \beta e_2$ *such that* $\alpha \neq 0, \beta \neq 0$ *in* F. *Now* $\alpha^{-1}\beta x_1 - x_2 = 0$ *is a non-trivial relation between* x_1 *and* x_2. *This shows that* x_1, x_2 *are linearly dependent. Hence the theorem is true for* $n = 1$.

Now, we assume the theorem for all vector spaces of dimension $\leq (n-1)$. *We shall prove it for a vector space of dimension* n. *For this, let* $B = \{e_1, e_2, \ldots, e_n\}$ *be a basis for* V. *Let* $x_1, x_2, \ldots, x_n, x_{n+1}$ *be any* $(n+1)$ *vectors of* V. *Then* $x_1 = a_{11}e_1 + a_{12}e_2 + \ldots + a_{1n}e_n$

$x_2 = a_{21}e_1 + a_{22}e_2 + \ldots + a_{2n}e_n$

$\vdots$

$x_n = a_{n1}e_1 + a_{n2}e_2 + \ldots + a_{nn}e_n$

$x_{n+1} = a_{(n+1)1}e_1 + a_{(n+1)2} + \ldots + a_{(n+1)n}e_n$

Now consider $x_2' = x_2 - {a_{11}}^{-1}a_{21}x_1$

$= (a_{22} - {a_{11}}^{-1}a_{21}a_{12})e_2 + \ldots + (a_{2n} - {a_{11}}^{-1}a_{21}a_{1n})e_n$

$x_3' = x_3 - {a_{11}}^{-1}a_{31}x_1$

$= (a_{32} - {a_{11}}^{-1}a_{31}a_{12})e_2 + \ldots + (a_{3n} - {a_{11}}^{-1}a_{31}a_{1n})e_n$

$\vdots$

$x_n' = x_n - {a_{11}}^{-1}a_{n1}x_1$

$= (a_{n2} - {a_{11}}^{-1}a_{n1}a_{12})e_2 + \ldots + (a_{nn} - {a_{11}}^{-1}a_{n1}a_{1n})e_n$

$x_{n+1}' = x_{n+1} - {a_{11}}^{-1}a_{(n+1)1}x_1$

$= (a_{(n+1)2} - {a_{11}}^{-1}a_{(n+1)1}a_{12})e_2 + \ldots (a_{(n+1)n} - {a_{11}}^{-1}a_{(n+1)1}a_{1(n+1)})e_n$

Let $W =< e_2, e_2, \ldots, e_n >$ *be a subspace generated by* $\{e_2, e_3, \ldots, e_n\}$. *Since every subset of a linearly independent set is linearly independent, so* W *is a* *subspace of V with dimension $n-1$.*

Therefore, by induction hypothesis the vectors $x'_2, x'_3, \ldots, x'_{(n+1)}$ are linearly dependent. This implies that there exists scalars $a_2, a_3, \ldots, a_{(n+1)}$ not all zero such that $a_2x'_2 + a_3x'_3 + \ldots + a_{(n+1)}x'_{(n+1)} = 0$.

Substutute the values of $x'_2, x'_3, \ldots, x'_{(n+1)}$, we get $a_2(x_2 - {a_{11}}^{-1}a_{21}x_1) + a_3(x_3 - {a_{11}}^{-1}a_{31}x_1) + \ldots + a_{(n+1)}(x_{n+1} - {a_{11}}^{-1}a_{(n+1)1}x_1) = 0$

$\Rightarrow -(a_2a_{11}^{-1}a_{21} + a_3a_{11}^{-1}a_{31} + \ldots a_{n+1}a_{11}^{-1}a_{(n+1)1})x_1 + a_2x_2 + \ldots + a_{n+1}x_{n+1} = 0$

which is a non-trivial relation among $x_1, x_2, \ldots, x_{n+1}$.

This shows that $x_1, x_2, \ldots, x_{n+1}$ are linearly dependent. Hence the theorem is true for all $n \in \mathbb{N}$.

(5.3.5) Theorem: *Let V be a vector space over F with $\dim(V) = n$. Then $B = \{x_1, x_2, \ldots, x_n\}$ is a basis of V if and only if B is linearly independent.*

Proof. *First, suppose that B is a basis of V. Then by definition it is linearly independent. conversly, suppose that B is linearly independent and $x \in V$ be any element. Then $x, x_1, x_2, \ldots, x_n$ are linearly dependent.*

$\Rightarrow$ there exists scalars $\alpha, \alpha_1, \ldots, \alpha_n$ not all zero such that $\alpha x + \alpha_1x_1 + \ldots + \alpha_nx_n = 0$. Here $\alpha \neq 0$, otherwise $x, x_1, x_2, \ldots, x_n$ will become linearly independent, which is not true.

Now, α^{-1} exists in $F \Rightarrow \alpha^{-1}(\alpha x + \alpha_1x_1 + \ldots + \alpha_nx_n) = 0$

$\Rightarrow x + \alpha^{-1}\alpha_1x_1 + \alpha^{-1}\alpha_2x_2 + \ldots + \alpha^{-1}\alpha_nx_n = 0$

$\Rightarrow x = -\alpha^{-1}\alpha_1x_1 - \alpha^{-1}\alpha_2x_2 - \ldots - \alpha^{-1}\alpha_nx_n$

$\Rightarrow x \in L(B)$.

This shows that $V \subset L(B)$ and hence $V = L(B)$. Thus B is a basis of V.

(5.3.6) Theorem (Existence Theorem): *Let V be a finite dimentional vector space over a field F. Then there exists a basis for V.*

OR There exists a basis for every vector space.

Proof Case I *If* $V = \{0\}$*, then* $B = \phi$ *is a basis of* V*.*

Case II *If* $V \neq \{0\}$*, then there exists* $0 \neq x_1 \in V$*. Consider* $B_1 = \{x_1\}$*. Then* B_1 *is a linearly independent subset of* V*.*

If $V = L(B_1)$*, then* B_1 *is a basis of* V*. If* $V \neq L(B_1)$*, then there exists* $x_2 \in V$ *such that* $x_2 \notin L(B_1)$*. Consider* $B_2 = \{x_1, x_2\}$*. Then* B_2 *is linearly independent subset of* V*. For this, suppose that* B_2 *is linearly dependent. Then there exists scalars* α, β *not both zero such that* $\alpha x_1 + \beta x_2 = 0$*. Here* $\beta \neq 0$*, otherwise* B_2 *is linearly independent. Now* β^{-1} *exists in* F *such that*

$\beta^{-1}(\alpha x_1 + \beta x_2) = 0$

$\Rightarrow x_2 = -(\beta^{-1}\alpha)x_1$

$\Rightarrow x_2 \in L(B_1)$*, a contradiction. Therefore* B_2 *is linearly independent.*

If $V = L(B_2)$*, then* B_2 *is a basis of* V*. If* $V \neq L(B_2)$*, then there exists* $x_3 \in V$ *such that* $x_3 \notin L(B_2)$*. Consider* $B_3 = \{x_1, x_2, x_3\}$*. Then* B_2 *is linearly independent subset of* V*. For this, suppose that* B_3 *is linearly dependent. Then there exists scalars* α, β, γ *not all zero such that* $\alpha x_1 + \beta x_2 + \gamma x_3 = 0$*. Here* $\gamma \neq 0$*, otherwise* B_3 *is linearly independent. Now* γ^{-1} *exists in* F *such that*

$\gamma^{-1}(\alpha x_1 + \beta x_2 + \gamma x_3) = 0$

$\Rightarrow x_3 = -(\gamma^{-1}\alpha)x_1 - (\gamma^{-1}\beta)x_2$

$\Rightarrow x_3 \in L(B_2)$*, a contradiction. Therefore* B_3 *is linearly independent.*

If $V = L(B_3)$*, then* B_3 *is a basis of* V*. If* $V \neq L(B_3)$*, then continuing the above process. Since* V *is finite dimensional vector space, so this process terminates after finite number of steps. That is untill we get a linearly independent subset* B *with* $\dim V$ *number of elements. Then* $L(B) = V$*.*

(5.3.7) Theorem (Extension Theorem)*: Let* V *be a vector space of dimension* n *and* $S = \{x_1, x_2, \ldots, x_m\}$ *be a linearly independent subset of* V*. Then* S *can be extended to form a basis of* V*.*

Proof. *If $m = n$, then there is nothing to prove. If $m < n$, then $L(S) \neq V$. This implies that there exists $x_{m+1} \in V$ such that $x_{m+1} \notin L(S)$. Cosider $S_1 = \{x_1, x_2, \ldots, x_m, x_{m+1}\}$. Then S_1 is linearly independent in V. For this, suppose S_1 is linearly dependent, then there exists scalars $a_1, a_2, \ldots, a_m, a_{m+1}$ not all zero such that $a_1x_1+a_2x_2+\ldots+a_mx_m+a_{m+1}x_{m+1} = 0$. Here, $a_{m+1} \neq 0$ otherwise S_1 is linearly independent which is contradiction to our assumption. Then there exists ${a_{m+1}}^{-1} \in F$ such that*

${a_{m+1}}^{-1}(a_1x_1 + a_2x_2 + \ldots + a_mx_m + a_{m+1}x_{m+1}) = 0$

$\Rightarrow {a_{m+1}}^{-1}a_1x_1 + {a_{m+1}}^{-1}a_2x_2 + \ldots + {a_{m+1}}^{-1}a_mx_m + x_{m+1} = 0$

$\Rightarrow -{a_{m+1}}^{-1}a_1x_1 - {a_{m+1}}^{-1}a_2x_2 - \ldots - {a_{m+1}}^{-1}a_mx_m = x_{m+1}$

This implies that $x_{m+1} \in L(S)$, a contradiction. Hence S_1 is linearly independent.

If $L(S_1) = V$, then S_1 is a basis of V. If $L(S_1) \neq V$, then there exists $x_{m+2} \in V$ such that $x_{m+2} \neq L(S_1)$. Cosider $S_2 = \{x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}\}$. Then as above, S_2 is linearly independent. Again, if $L(S_2) = V$, then S_2 is a basis of V, otherwise, repeat the above process. But this process terminates after $(n - m)$ steps as the dimension of V is n. That is, we get a linearly independent set $B = \{x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots, x_n\}$ such that $L(B) = V$. Hence there exists a basis for every finite dimensional vector spaces.

(5.3.8) Theorem*: Let V be a finite dimensional vector space over F. Then any two bases of V have the same number of elements.*

Proof. *Let $B_1 = \{x_1, x_2, \ldots, x_m\}$ and $B_2 = \{y_1, y_2, \ldots, y_n\}$ be any two bases of V.*

Case I *When B_1 is linearly independent and B_2 is a basis: In this case number of elements of B_1 can not excceed number of elements of B_2 because any $n+1$ elements of V are L.D. This implies that $m \leq n$.....(1)*

Case II *When B_2 is linearly independent and B_1 is basis: In this case number of elements of B_2 can not excceed number of elements of B_1 because any $m+1$ elements of V are L.D. This implies that $n \leq m$.....(2)*
From (1) and (2), we have $m = n$. Hence any two bases of a vector space have same number of elements.
(5.3.9) Theorem*: Let V_1 amd V_2 be two subspaces of a finite dimensional vector space $V(F)$. Then*

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2).$$

Proof. *Let $B_0 = \{x_1, x_2, \ldots, x_k\}$ be a basis of $V_1 \cap V_2$. Then B_0 is a linearly independent subset of V_1 and V_2. Therefore, by basis extension theorem, B_0 can be extended to form bases of V_1 and V_2. Let $B_1 = \{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_m\}$ and $B_2 = \{x_1, x_2, \ldots, x_k, x'_{k+1}, \ldots, x'_n\}$ be bases of V_1 and V_2 respectively. That is,* $\dim V_1 = m, \dim V_2 = n$ *and* $\dim(V_1 \cap V_2) = k$.
Claim: $B = \{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_m, x'_{k+1}, \ldots, x'_n\}$ is a basis of $V_1 + V_2$.
We first check that B is linearly independent,
for this, consider $a_1x_1 + a_2x_2 + \ldots + a_kx_k + a_{k+1}x_{k+1} + \ldots + a_mx_m + a'_{k+1}x'_{k+1} + \ldots + a'_nx'_n = 0$........(1)
$\Rightarrow a_1x_1 + a_2x_2 + \ldots + a_kx_k + a_{k+1}x_{k+1} + \ldots + a_mx_m = -a'_{k+1}x'_{k+1} - \ldots - a'_nx'_n \in V_2$
$\Rightarrow a_1x_1 + a_2x_2 + \ldots + a_kx_k + a_{k+1}x_{k+1} + \ldots + a_mx_m \in V_1 \cap V_2$
$\Rightarrow a_1x_1 + a_2x_2 + \ldots + a_kx_k + a_{k+1}x_{k+1} + \ldots + a_mx_m = b_1x_1 + \ldots + b_kx_k$
$\Rightarrow (a_1 - b_1)x_1 + (a_2 - b_2)x_2 + \ldots + (a_k - b_k)x_k + a_{k+1}x_{k+1} + \ldots + a_mx_m = 0$
$\Rightarrow a_{k+1} = \ldots = a_m = 0$ *as B_1 is linearly independent.*
Put these values in (1), we get $a_1x_1 + a_2x_2 + \ldots + a_kx_k + a'_{k+1}x_{k+1} + \ldots + a_nx'_n = 0$
$\Rightarrow a_1 = a_2 = \ldots = a_k = a'_{k+1} = \ldots = a'_n = 0$ *as B_2 is linearly independent.*
Hence B is linearly independent.

Now, let $x \in V_1 + V_2$. *Then* $x = y + z$, *where* $y \in V_1$ *and* $z \in V_2$. *Therefore,* $y = a_1x_1 + a_2x_2 + \ldots + a_kx_k + a_{k+1}x_{k+1} + \ldots + a_mx_m$ *and* $z = b_1x_1 + b_2x_2 + \ldots + b_kx_k + a'_{k+1}x'_{k+1} + a'_nx'_n$.

Now $x = a_1x_1 + a_2x_2 + \ldots + a_kx_k + a_{k+1}x_{k+1} + a_mx_m + b_1x_1 + b_2x_2 + \ldots + b_kx_k + a'_{k+1}x'_{k+1} + a'_nx'_n$

$\Rightarrow x = (a_1 + b_1)x_1 + (a_2 + b_2)x_2 + \ldots + (a_k + b_k)x_k + a_{k+1}x_{k+1} + a_mx_m + a'_{k+1}x'_{k+1} + a'_nx'_n$

$\Rightarrow x \in L(B)$. *This implies that*

$$V = L(B).$$

Hence B *is a basis of* $V_1 + V_2$

and $\dim(V_1 + V_2) = k + (m - k) + (n - k) = n + m - k$

$\Rightarrow \dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$.

(5.3.10) Theorem: *Let* V *be a finite dimensional vector space over a field* F *and* W *be its subspace. Then*

$$\dim(V/W) = \dim V - \dim W.$$

Proof. *Let* $B_1 = \{x_1, x_2, \ldots, x_k\}$ *be a basis of* W. *Then* B_1 *is a linearly independent subset of* V. *Now, by extension theorem* B_1 *can be extended to form a basis of* V. *Let* $B_2 = \{x_1, x_2, \ldots, x_k, x_{k+1, \ldots, x_n}\}$ *be a basis of* V.

Claim: $B = \{x_{k+1} + W, x_{k+2} + W, \ldots, x_n + W\}$ *is a basis of* V/W.

First , we shall prove that B *is linearly independent.*

For this, consider $a_{k+1}(x_{k+1} + W) + \ldots + a_n(x_n + W) = 0 + W$

$\Rightarrow (a_{k+1}x_{k+1}) + W + \ldots + (a_nx_n) + W = W$

$\Rightarrow (a_{k+1}x_{k+1} + \ldots + a_nx_n) + W = W$

$\Rightarrow (a_{k+1}x_{k+1} + \ldots + a_nx_n) \in W$

$\Rightarrow a_{k+1}x_{k+1} + \ldots + a_nx_n = a_1x_1 + a_2x_2 + \ldots + a_kx_k$

$\Rightarrow a_1x_1 + a_2x_2 + \ldots + a_kx_k - a_{k+1}x_{k+1} - \ldots - a_nx_n = 0$

$\Rightarrow a_1 = a_2 = \ldots = a_n = 0.$

Hence B is linearly independent.

Now, let $x + W \in V/W$ be any element. Then $x \in V \Rightarrow x = a_1x_1 + a_2x_2 + \ldots + a_kx_k + a_{k+1}x_{k+1} + \ldots + a_nx_n$

$\Rightarrow x + W = (a_1x_1 + a_2x_2 + \ldots + a_kx_k + a_{k+1}x_{k+1} + \ldots + a_nx_n) + W$

$\Rightarrow x + W = (a_1x_1 + a_2x_2 + \ldots + a_kx_k + a_{k+1}x_{k+1} + \ldots + a_nx_n) + W$

$\Rightarrow x + W = (a_1x_1 + a_2x_2 + \ldots + a_kx_k) + W + (a_{k+1}x_{k+1} + \ldots + a_nx_n) + W$

$= W + (a_{k+1}x_{k+1} + \ldots + a_nx_n) + W$

$= (a_{k+1}x_{k+1} + \ldots + a_nx_n) + W$

$= (a_{k+1}x_{k+1}) + W + \ldots + (a_nx_n) + W$

$= a_{k+1}(x_{k+1} + W) + \ldots + a_n(x_n + W) \in L(B)$

$\Rightarrow x + W \in L(B)$

$$\Rightarrow V/W = L(B).$$

Hence B is a basis of V and

$$\dim(V/W) = n - k = \dim V - \dim W.$$

(5.4) Examples

1. *Let $V = \mathbb{R}^n$ be a vector space over $\mathbb{R}$. Then $B = \{e_1 = (1, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 1)\}$ is a basis of V.*

Let's first show that B is linearly independent. For, suppose that $a_1(1, \ldots, 0) + \ldots + a_n(0, \ldots, 1) = (0, 0, \ldots, 0)$

$\Rightarrow (a_1, \ldots, a_n) = (0, 0, \ldots, 0) \Rightarrow a_1 = a_2 = \ldots = a_n = 0$. *Therefore $\{e_1, \ldots, e_n\}$ are linearly independent.*

Now, we know that $L(B) \subset V$. Let $x \in V \Rightarrow x = (x_1, x_2, \ldots, x_n)$, for $x_i \in \mathbb{R}, \forall i$

$\Rightarrow x = x_1(1, 0 \ldots, 0) + \ldots + x_n(0, 0, \ldots, 1)$

$\Rightarrow x \in L(B)$

$\Rightarrow V \subset L(B)$.

Thus, $V = L(B)$ *and* B *is a basis of* V. *Also,* $\dim(V) = n$.

2. *Let* $V = F[x]$ *be a vector space of polynomials. Then* $B = \{f_k(x) = x^k | k = 0, 1, 2, \ldots\}$ *forms a basis for* V.

Solution *First,* B *is L.I.*

For this we shall show that every finite subset of B *is linearly independent i.e* $\{f_0, f_1, \ldots, f_n\}$ *is linearly independent*

$a_0 f_0 + \ldots + a_n f_n = 0$

$\Rightarrow a_0 + a_1 x + \ldots + a_n x^n = 0$

$\Rightarrow a_0 = a_1 = \ldots a_n = 0$.

Every $f(x) \in F[x]$ *can be written as* $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$

$\Rightarrow f(x) \in L(B)$. *Hence* B *is a basis of* V *and* $\dim(V) = \infty$.

3. *Let* $V = F_n[x]$ *be a vector space of polynomials of degree* $\leq n$. *Then* $B = \{f_k(x) = x^k | k = 0, 1, 2, \ldots, n\}$ *forms a basis for* V.

Solution. *First, we see that* B *is linearly independent.*

For this, we consider $a_1(1) + a_2(x) + \ldots + a_n x^n = 0 = 0(1) + 0x + \ldots + 0x^n$

$\Rightarrow a_1 = 0 = \ldots = a_n$.

This shows that B *is linearly independent.*

Now, $L(S) \subset V$. *Let* $f(x) \in V$. *Then* $f(x) = a_0 + a_1 x + \ldots + a_m x^m$ *and* $m \leq n$

$\Rightarrow f(x) = a_0 + a_1 x + \ldots + a_m x^m + 0x^{m+1} + \cdots + 0x^n$

$\Rightarrow f(x) \in L(S)$

$\Rightarrow V \subset L(S)$.

Hence $V = L(S)$. *The* $\dim(V) = n + 1$.

4. *Show that the vectors* $(1, 1, 1)$, $(1, 0, 1)$ *and* $(1, -1, -1)$ *of* $\mathbb{R}^3$ *form a basis of* $\mathbb{R}^3$.

Solution *To show that* $B = (1, 1, 1), (1, 0, 1), (1, -1, -1)$ *form a basis of* V*, it is enough to check that* B *is linearly independent. For this, consider*

$a(1, 1, 1) + b(1, 0, 1) + c(1, -1, -1) = (0, 0, 0)$

$\Rightarrow (a + b + c,\ a - c,\ a + b - c) = (0, 0, 0)$

$a + b + c = 0$

$a - c = 0$

$a + b - c = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$\textit{then} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 2 \neq 0$$

$\Rightarrow a = b = c = 0$. *Hence,* B *is a basis of* V.

(5.5) Let Us Sum Up: *In order to understand the vector space structure, it is enough to understand its basis. So basis of vector space is the integral unit of vector space. In this lesson we have defined basis and dimension of a vector space and illustrated these notions with examples. With the existence theorem, extension theorem, we have observed that every vector space has a basis and every linearly independent subset of it can be extended to form a basis.*

(5.6) **Lesson End Excercise**

1. *Examine whether the following set of vectors in* $\mathbb{R}^3$ *form a basis or not:*

(i) $(1, 0, -1), (1, 2, 1), (0, -3, 2)$

(ii) $(1, 1, 1), (1, 2, 3), (-1, 0, 1)$

(iii) $(1, \frac{2}{5}, -1), (0, 1, 2), (\frac{3}{4}, -1, 1)$

(iv) $(1, 0, 0), (0, 1, 0), (1, 1, 0), (1, 1, 1)$.

2. *Let* $V = \{[a_{ij}] : a_{ij} \in \mathbb{R}, 1 \leq i, j \leq 2\}$. *Then show that the set* $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ *form a basis of* V.

3. *Show that the dimension of the vector space* $\mathbb{Q}(\sqrt{3})$ *over* $\mathbb{Q}$ *is* 2.

4. *Extend* $B = \{(1, 2, 5)\}$ *to form a basis of* $\mathbb{R}^3$.

Hint: *Since* $(1, 0, 0) \notin L(B)$ *as* $L(B) = \{(\alpha, 2\alpha, 5\alpha) : \alpha \in F\}$. $\Rightarrow B_1 = \{(1, 2, 5), (1, 0, 0)\}$ *is linearly independent. Also,* $(0, 1, 0) \notin L(B_1)$. *Therefore,* $B_2 = \{(1, 2, 5), (1, 0, 0), (0, 1, 0)\}$ *is linearly independent. Hence* B_2 *is an exended basis of* $\mathbb{R}^3$.

5.*Let* $V = \{f(x) \in \mathbb{R}[x] | \deg(f(x)) \leq 3\}$ *be a vector space of polynomials over* $\mathbb{R}$. *Show that* $\dim(V) = 4$.

(5.7) University Model Questions

1. *Define a basis of a vecor space* $V(F)$. *Show that* $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ *is a basis of* $\mathbb{R}^3$. *Find a basis of* $\mathbb{R}^3$ *different from* B.

2. *Let* V *be a finite dimensional vector space over* F *and* W *be its subspace.Prove that* $\dim(V/W) = \dim V - \dim W$.

3. *Let* V_1 *and* V_2 *be two subspaces of a finite dimensional vector space* V. *Show that* $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$.

(5.8) Suggested Readings *:(i) N.S. Gopalakrishnan, University Algebra, New Age International (P) Limited, Publishers.*

(ii) Kenneth Hoffman, Ray Kunze, Linear Algebra,Prentice Hall India.

(6.1) Introduction *: As we are familiar with the notion of isomorphism of groups and isomorphism of rings, we can also define isomorphism between two vector spaces. In the definition of isomorphism between two vector spaces, we assume the both the vector spaces over the same field. A vector space homomorphism is also called as a linear transformation .*

(6.2) Objectives*: (i) In this lesson, students will learn the algebraically same vector spaces upto isomorphism*

(ii) they will learn how two vector spaces can be differentiated.

(6.3) Isomorphic vector spaces

(6.3.1) Definition: Vector Space Homomorphism or Linear Transformation*: Let V and W be vector spaces over the field F. Then a mapping*

$T : V \to W$ is said to be a linear transformation or vector space homomorphism if

(i) $T(x + y) = T(x) + T(y), \forall\ x, y \in V$

and (ii) $T(\alpha x) = \alpha T(x), \forall\ x \in V\ , \alpha \in F$.

A linear transformation $T : V(F) \to W(F)$ is said to be an **isomorphism** *if T is one-one and onto. The vector spaces V and W are said to be isomorphic if there exists an isomorphism between them and can be written as $V \cong W$.*

(6.3.2) Theorem *Let $T : V(F) \to W(F)$ be a linear transformation. Then (i) $T(0) = 0'$(ii) $T(-x) = -T(x)$(iii) $T(x - y) = T(x) - T(y), \forall x, y \in V$.*

Proof *(i) We have $0 + 0 = 0 \Rightarrow T(0 + 0) = T(0)$*

$\Rightarrow T(0) + T(0) = T(0) + 0'$

$\Rightarrow T(0) = 0'$.

(ii) We have $x + (-x) = 0$

$\Rightarrow T(x + (-x)) = T(0) \Rightarrow T(x) + T(-x) = 0'$

$\Rightarrow T(x) = -T(x)$.

(iii) $T(x - y) = T(x + (-y)) = T(x) + T(-y) = T(x) - T(y)$ (by (ii))

(6.3.3) Theorem *Let V be a vector space over a field F with $\dim V = n$. Then $V \cong F^n$.*

proof. *Let $B = \{x_1, x_2, \ldots, x_n\}$ be a basis of V. Then every element x of V can be uniquely written as*

$$x = \sum_{i=1}^{n} a_i x_i.$$

Now define a rule $T : V \to F^n$ by

$$T(\sum_{i=1}^{n} a_i x_i) = (a_1, a_2, \ldots, a_n).$$

(I) T **is well-defined function***: Let $x = \sum_{i=1}^{n} a_i x_i$ and $x = \sum_{i=1}^{n} b_i x_i$. Then $\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} b_i x_i$*

$\Rightarrow \sum_{i=1}^{n}(a_i - b_i)x_i = 0$

$\Rightarrow a_i - b_i = 0, \; \forall \; i$

$\Rightarrow a_i = b_i, \; \forall \; i$

$\Rightarrow (a_1, \, a_2, \, \ldots, \, a_n) = (b_1, \, b_2, \, \ldots, \, b_n)$

$\Rightarrow T(\sum_{i=1}^{n} a_i x_i) = T(\sum_{i=1}^{n} b_i x_i)$.

This implies that T is a well-defined function.

(II) T **is linear transformation***:*

$$\begin{aligned}
T(x+y) &= T(\sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i x_i) \\
&= T(\sum_{i=1}^{n} (a_i + b_i) x_i) \\
&= (a_1 + b_1, \, a_2 + b_2, \, \ldots, \, a_n + b_n) \\
&= (a_1, \, a_2, \, \ldots, \, a_n) + (b_1, \, b_2, \, \ldots, \, b_n) \\
&= T(\sum_{i=1}^{n} a_i x_i) + T(\sum_{i=1}^{n} b_i x_i) \\
&= T(x) + T(y)
\end{aligned}$$

Also,

$$\begin{aligned}
T(\alpha x) &= T(\alpha \sum_{i=1}^{n} a_i x_i) \\
&= T(\sum_{i=1}^{n} \alpha a_i x_i) \\
&= (\alpha a_1, \, \alpha a_2, \, \ldots, \, \alpha a_n) \\
&= \alpha (a_1, \, a_2, \, \ldots, \, a_n) \\
&= \alpha T(\sum_{i=1}^{n} a_i x_i) \\
&= \alpha T(x)
\end{aligned}$$

Therefore, T is a linear transformation.

(III) T **is one-one**: *Let*

$$\begin{aligned} T(\sum_{i=1}^{n} a_i x_i) &= T(\sum_{i=1}^{n} b_i x_i) \\ (a_1, a_2, \ldots, a_n) &= (b_1, b_2, \ldots, b_n) \end{aligned}$$

$$\Rightarrow a_i = b_i, \forall\, i \Rightarrow \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} b_i x_i.$$

Hence T is one-one.

(IV) T **is onto**: *For each $(a_1, a_2, \ldots, a_n) \in F^n$, there exists $x = \sum_{i=1}^{n} a_i x_i \in V$ such that $T(\sum_{i=1}^{n} a_i x_i) = (a_1, a_2, \ldots, a_n)$.*

Hence, T is an isomorphism and $V \cong F^n$.

(6.3.4) Theorem *Let V and W be finite dimensional vector spaces over F. Then $\dim(V) = \dim(W)$ if and only if $V \cong W$.*

Proof *Let us first suppose that $\dim(V) = \dim(W) = n(say)$. Let $B_1 = \{x_1, x_2, \ldots, x_n\}$ and $B_2 = \{y_1, y_2, \ldots, y_n\}$ be bases of V and W respectively. Define a rule $T : V \to W$ by*

$$T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i y_i.$$

Then T is a well-defined function as every element in V as well as in W has the unique representation.

Now, T **is linear transformation**:

$$\begin{aligned} T(\sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i x_i) &= T(\sum_{i=1}^{n} (a_i + b_i) x_i) \\ &= \sum_{i=1}^{n} (a_i + b_i) y_i \\ &= \sum_{i=1}^{n} a_i y_i + \sum_{i=1}^{n} b_i y_i \\ &= T(\sum_{i=1}^{n} a_i x_i) + T(\sum_{i=1}^{n} b_i x_i) \end{aligned}$$

Also,

$$\begin{aligned} T(\alpha\sum_{i=1}^{n} a_i x_i) &= T(\sum_{i=1}^{n} \alpha a_i x_i) \\ &= \sum_{i=1}^{n} \alpha a_i y_i \\ &= \alpha\sum_{i=1}^{n} a_i y_i \\ &= \alpha T(\sum_{i=1}^{n} a_i x_i) \end{aligned}$$

T **is one-one**: *Let*

$$\begin{aligned} T(\sum_{i=1}^{n} a_i x_i) &= T(\sum_{i=1}^{n} b_i x_i) \\ \sum_{i=1}^{n} a_i y_i &= \sum_{i=1}^{n} b_i y_i \\ \sum_{i=1}^{n}(a_i - b_i) y_i &= 0 \\ a_i - b_i &= 0 \\ a_i &= b_i,\ \forall i \\ \sum_{i=1}^{n} a_i x_i &= \sum_{i=1}^{n} b_i x_i \end{aligned}$$

T **is onto**: *For each $y = \sum_{i=1}^{n} a_i y_i \in W$, there exists $x = \sum_{i=1}^{n} a_i x_i \in V$ such that $T(x) = T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i y_i$. Hence T is an isomorphism and $V \cong W$.*

Conversely, suppose that $V \cong W$ and $T : V \to W$ is an isomorphism. To show that $\dim V = \dim W$, we need to show that the bases of V and W have the same number of elements. For this, let $B = \{x_1, x_2, \ldots, x_n\}$ be a basis of V.

Claim: $B' = \{T(x_1), T(x_2), \ldots, T(x_n)\}$ is a basis of W.

For, consider $a_1T(x_1) + a_2T(x_2) + \ldots + a_nT(x_n) = 0'$

$\Rightarrow T(a_1x_1) + T(a_2x_2) + \ldots + T(a_nx_n) = 0'$

$\Rightarrow T(a_1x_1 + a_2x_2 + \ldots + a_nx_n) = T(0)$

$\Rightarrow a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$

$\Rightarrow a_1 = a_2 = \ldots = a_n = 0$ *as* B *is linearly independent.*

Now, let $y \in W$ *be any element. Then there exist* $x \in V$ *such that* $T(x) = y$.

This implies that $x = \sum_{i=1}^{n} a_ix_i$ *and* $y = T(x) = T(\sum_{i=1}^{n} a_ix_i)$

$= \sum_{i=1}^{n} a_iT(x_i)$

$\Rightarrow y \in L(B') \Rightarrow V = L(B')$.

Hence B' *is a basis of* W *and* $\dim W = n = \dim V$.

(6.3.5) Theorem *Let* V *and* W *be vector spaces over a field* F. *Then a mapping* $T : V \to W$ *is a linear transformation if and only if* $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y), \forall\ \alpha, \beta \in F$ *and* $x,\ y\ \in V$.

Proof. *Suppose that* $T : V \to W$ *is a linear transformation. Then* $T(\alpha x + \beta y) = T(\alpha x) + T(\beta y)$

$= \alpha T(x) + \beta T(y)$

$\Rightarrow T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$.

Conversely, suppose that $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

Put $\alpha = 1 = \beta$, *the we get* $T(x + y) = x + y$.

Now put $\beta = 0$, *we get* $T(\alpha x) = T(\alpha x + 0y) = \alpha T(x) + 0T(y)$

$= \alpha T(x)$.

Therefore T *is a linear transformation.*

(6.4) Examples

1. *The mappings* $O : V \to W$ *and* $I : V \to V$ *defined by*

$$O(x) = 0, \forall\ x \in V$$

and

$$I(x) = x, \forall\ x\ \in V$$

respectively are linear transformations.

For these, we have $O(x + y) = 0 = 0 + 0 = O(x) + O(y)$ *and* $O(\alpha x) = 0 = \alpha 0 = \alpha O(x)$.

Similarly, $I(x + y) = x + y = I(x) + I(y), \forall\ x, y \in V$ *and* $I(\alpha x) = \alpha x = \alpha I(x), \forall \alpha \in F$.

2. *Let* $V = F^n$ *be a vector space over the field* F. *Then the mapping defined by* $T((a_1, a_2, \ldots, a_n)) = a_1, \forall\ a_i \in F, i = 1, 2, \ldots, n$ *is a linear transformation.*

For this, $T((a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n))$

$= T(a_1 + b_1, \ldots, a_n + b_n) = T((a_1 + b_1, \ldots, a_n + b_n)) = a_1 + b_1$

$= T(a_1, a_2, \ldots, a_n) + T(b_1, b_2, \ldots, b_n)$

and

$$\begin{aligned} T(\alpha(a_1, a_2, \ldots, a_n)) &= T(\alpha a_1, \alpha a_2, \ldots, \alpha a_n) \\ &= \alpha a_1 \\ &= \alpha T(a_1, a_2, \ldots, a_n). \end{aligned}$$

3. *Let* $V = F[x]$ *be the vector space of polynomials over a field* F *and* $V' = F$. *Then the mapping* $T : V \to V'$ *defined by* $T(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n) = a_0$ *is a linear transformation. For this,*

$T((a_0 + a_1x + a_2x^2 + \ldots + a_nx^n) + (b_0 + b_1x + b_2x^2 + \ldots + b_nx^n)) = T((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots + (a_n + b_n)x^n)$

$= a_0 + b_0$

$= T(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n) + T(b_0 + b_1x + b_2x^2 + \ldots + b_nx^n)$.

Now,
$$\begin{aligned} T(\alpha(a_0 + a_1x + \ldots + a_nx^n)) &= T(\alpha a_0 + \alpha a_1x + \ldots + \alpha a_nx^n) \\ &= \alpha a_0 \\ &= \alpha T(a_0 + a_1x + \ldots + a_nx^n). \end{aligned}$$

4. *Let* $V = C[0,\,1]$ *be the space of real valued continuous functions on* $[0,\,1]$ *and* $V' = \mathbb{R}$. *Then mapping* $T : V \to V'$ *defined by* $T(x(t)) = x(\frac{1}{2}),\ \forall\ x \in V$ *is a linear transformation.*

For this, we have $T(x(t) + y(t)) = T((x+y)(t)) = (x+y)(\frac{1}{2})$

$$= x(\tfrac{1}{2}) + y(\tfrac{1}{2})$$

$$= T(x(t)) + T(y(t)).$$

Similarly, $T(\alpha(x(t))) = T(\alpha x(t))$

$$= \alpha x(\tfrac{1}{2})$$

$$= \alpha T(x(t)),\ \forall x \in V \text{ and } \forall \alpha \in F.$$

5. *Let* $V = \mathbb{C}$ *and* $V' = \mathbb{R}^2$ *be vector spaces over* $\mathbb{R}$. *Then the mapping* $T : V \to V'$ *defined by* $T(a + \imath b) = (a,\, b)$ *is an isomorphism.*

Solution *First we shall show that* T *is a linear transformation. For this,*

$$T((a + \imath b) + (c + \imath d)) = T((a + c) + \imath(b + d))$$

$$= (a + c,\, b + d)$$

$$= (a,\, b) + (c,\, d)$$

$$= T(a + \imath b) + T(c + \imath d).$$

Now, $T(\alpha(a + \imath b)) = T(\alpha a + \imath \alpha b)$

$$= (\alpha a,\, \alpha b)$$

$$= \alpha(a,\, b)$$

$$= \alpha T(a + \imath b).$$

Now, T **is one-one**

For this, let $T(a + \imath b) = T(c + \imath d)$

$\Rightarrow (a,\, b) = (c,\, d)$

$\Rightarrow a = c$ *and* $b = d$

$\Rightarrow a + \imath b = c + \imath d.$

Also, T *is onto. For this, for each* $(x,\, y) \in \mathbb{R}^2$ *there exists* $x + \imath y \in \mathbb{C}$ *such* *that* $T(x + \iota y) = (x, y)$. *Hence* T *is an isomorphism and* $\mathbb{C} \cong \mathbb{R}^2$.

6. *Let* $T_1 : V \to W$ *and* $T_2 : V \to W$ *be two linear transformations. Then (i) the mapping* $T_1 + T_2 : V \to W$ *defined by* $(T_1 + T_2)(x) = T_1(x) + T_2(x), \forall\, x \in V$ *is a linear transformation and*

(ii) $(\alpha T)(x) = \alpha T(x)$ *is also a linear transformation.*

Proof *First,* $(T_1 + T_2)(x + y) = T_1(x + y) + T_2(x + y)$

$= T_1(x) + T_1(y) + T_2(x) + T_2(y)$

$= (T_1(x) + T_2(x)) + (T_1(y) + T_2(y))$

$= (T_1 + T_2)(x) + (T_1 + T_2)(y)$.

Now, let $\alpha \in F$ *and* $x \in V$,

then $(T_1 + T_2)(\alpha x) = T_1(\alpha x) + T_2(\alpha x)$

$= \alpha T_1(x) + \alpha T_2(x)$

$= \alpha(T_1(x) + T_2(x)) = \alpha(T_1 + T_2)(x)$.

This shows that $T_1 + T_2$ *is a linear transformation.*

(ii)$(\alpha T)(x + y) = \alpha T(x + y) = \alpha(T(x) + T(y))$

$= \alpha T(x) + \alpha T(y) = (\alpha T)(x) + (\alpha T)(y)$

$= (\alpha T)(x + y)$.

Similarly $(\alpha T)(ax) = \alpha T(ax) = \alpha a T(x) = a\alpha T(x) = a(\alpha T)(x)$

$\Rightarrow \alpha T$ *is a linear transformation.*

(6.5) Finite dimensional and infinite dimensional vector spaces

(6.5.1) Definition *A vector space* V *over a field* F *is said to be a finite dimensional if* $\dim(V) < \infty$ *and* V *is said to be an infinite dimensional if* $\dim(V)$ *is not finite.*

(6.5.2) Example *The vector space* $\mathbb{R}$ *over* $\mathbb{Q}$ *is not finite dimensional.*

Solution *Suppose that* $\mathbb{R}$ *is a finite dimensional vector space over* $\mathbb{Q}$. *Then*

$$\mathbb{R} \cong Q^n$$

which in not true because $\mathbb{Q}^n$ *is countable set whereas* $\mathbb{R}$ *is uncountable. Therefore,* $\mathbb{R}$ *is an infinte dimensional vector space over* $\mathbb{Q}$.

(6.6) Let Us Sum Up *In this lesson, we defined vector space homomorphism and vector space isomorphism, then explored various properties of homomorphism. Then illustrated this notion with the help of examples. In the end, the finite and infinite dimensional vector spaces have also been discussed.*

(6.7) **Lesson End Exercise**

1. *Let* $V = F^{n+1}$ *and* $W = F_n[x]$, *the space of all polynomials of degree atmost* n. *Define the mapping* $T : V \to W$ *by* $T(a_0, a_1, \ldots, a_n) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$. *Show that* T *is an isomorphism.*

2. *Let* $V = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ *and* $V' = \{a + ib \mid a, b \in \mathbb{R}\}$ *be vector spaces over* $\mathbb{R}$. *Then show that* $V \cong V'$.

3. *Let* $V = \{(x_1, x_2, 0) | x_1, x_2 \in \mathbb{R}\}$ *be a subspace of* $\mathbb{R}^3$. *Then show that* $V \cong \mathbb{R}^2$.

4. *Let* V *and* W *be two vector spaces over a field* F. *Then the set* $L(V, W)$ *of all linear transformations of* V *in* W *forms a vector space over* F *under the operations* $+$ *and scalar multiplication defined as*

$$(T_1 + T_2)(x) = T_1(x) + T_2(x) \text{ and } (\alpha T)(x) = \alpha T(x)$$

respectively.

Solution. *Properties under* $+$:

(i) Let $T_1, T_2 \in L(V, W)$*. Then* $T_1(\alpha x + \beta y) = \alpha T_1(x) + \beta T_1(y)$ *and* $T_2(\alpha x + \beta y) = \alpha T_2(x) + \beta T_2(y) \, \forall \, x, y \in \; V$ *and* $\forall \, \alpha, \beta \in F$ *Now,* $(T_1 + T_2)(\alpha x + \beta y) = T_1(\alpha x + \beta y) + T_2(\alpha x + \beta y)$

$$= \alpha T_1(x) + \beta T_1(y) + \alpha T_2(x) + \beta T_2(y)$$

$$= \alpha(T_1(x) + T_2(x)) + \beta(T_1(y) + T_2(y))$$

$$= \alpha(T_1 + T_2)(x) + \beta(T_1 + T_2)(y)$$

$\Rightarrow (T_1 + T_2)(\alpha x + \beta y) = \alpha(T_1 + T_2)(x) + \beta(T_1 + T_2)(y)$.

(ii) Let $T_1, T_2, T_3 \; \in \; L(V, W)$.

Then $((T_1 + T_2) + T_3)(x) = (T_1 + T_2)(x) + T_3(x)$

$$= (T_1(x) + T_2(x)) + T_3(x)$$

$$= T_1(x) + (T_2(x) + T_3(x))$$

$$= T_1(x) + (T_2 + T_3)(x)$$

$$= (T_1 + (T_2 + T_3))(x)$$

$\Rightarrow (T_1 + T_2) + T_3 = T_1 + (T_2 + T_3)$.

(iii) Define $O : V \to W$ *by* $O(x) = 0, \forall \; x$ *in* V.

Then $O(\alpha x + \beta y) = 0 = \alpha 0 + \beta 0 = \alpha O(x) + \beta O(x)$

$\Rightarrow O \in L(V, W)$ *such that*

$(T + O)(x) = O(x) + T(x) = 0 + T(x) = T(x), \forall \; T \; \in L(V, W)$.

(iv) For each $T \in L(V, W)$*, define* $-T : V \to W$ *by* $(-T)(x) = -T(x)$*. Then*

$(-T)(\alpha x + \beta y) = -T(\alpha x + \beta y)$

$= -(\alpha T(x) + \beta T(y)) = -\alpha T(x) - \beta T(y)$

$= \alpha(-T(x) + \beta(-T(y))) = \alpha(-T)(x) + \beta(-T)(y)$

$\Rightarrow -T \in L(V, W)$ *such that* $(-T + T)(x) = -T(x) + T(x) = 0$.

(v) Let $T_1, T_2 \in L(V, W)$*. Then* $(T_1 + T_2)(x) = T_1(x) + T_2(x) = T_2(x) + T_1(x) = (T_2 + T_1)(x)$

$\Rightarrow T_1 + T_2 = T_2 + T_1$*. Thus* $L(V, W)$ *is an abelian group under* $+$.

Properties under scalar multiplication: *Let $\alpha, \beta \in F$ and $T, T_1, T_2 \in L(V, W)$. Then*

$$\begin{aligned}
(i)\ [(\alpha+\beta)T](x) &= (\alpha+\beta)T(x) \\
&= \alpha T(x) + \beta T(x) \\
&= (\alpha T)(x) + (\beta T)(x) \\
&= (\alpha T + \beta T)(x) \\
\Rightarrow (\alpha+\beta)T &= \alpha T + \beta T.
\end{aligned}$$

$$\begin{aligned}
(ii)\ [\alpha(T_1+T_2)](x) &= \alpha(T_1(x)+T_2(x)) \\
&= \alpha T_1(x) + \alpha T_2(x) \\
&= (\alpha T_1)(x) + (\alpha T_2)(x) \\
&= (\alpha T_1 + \alpha T_2)(x) \\
\Rightarrow \alpha(T_1+T_2) &= \alpha T_1 + \alpha T_2.
\end{aligned}$$

$$\begin{aligned}
(iii)\ [(\alpha\beta)T](x) &= (\alpha\beta)T(x) \\
&= \alpha(\beta T(x)) \\
&= \alpha(\beta T)(x) \\
\Rightarrow (\alpha\beta)T &= \alpha(\beta T).
\end{aligned}$$

(iv) $(1T)(x) = 1T(x) = T(x) \Rightarrow 1T = T$.

This shows that $L(V, W)$ is a vector space.

(6.8) University Model Questions

1. *Prove that the subset of matrices* $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ *in $M_2(F)$ for all $a \in F$ field is a vector space over F, which is isomorphic to the field F.*

2. *Show that the linear transformation* $T : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $T(x, y) = x$ *is onto but not one-one.*

3. *Show that the linear transformation* $T : \mathbb{R}^2 \to \mathbb{R}^3$ *defined by* $T(x, y) = (x, x - y, x + y)$ *is one-one but not onto.*

(6.9) Suggested Readings *:(i) N.S. Gopalakrishnan, University Algebra, New Age International (P) Limited, Publishers.*

(ii) Kenneth Hoffman, Ray Kunze, Linear Algebra,Prentice Hall India.

(7.1) Introduction: *As we have proved in the exercises of previous lesson that $L(V, W)$ the set of all linear transformations from a vecor space $V(F)$ to $W(F)$ is a vecor space over the field F. Here in this lesson we will study its particular part i.e $L(V, F)$ the set of all linear functionals on V which turns out to be a vector space over F.*

(7.2) Objectives *In this lesson, we define the dual space of a vector space and illustrate with Examples.*

(7.3) Linear functional

(7.3.1) Definition: *A linear transformation $f : V \to F$ from a vecor space $V(F)$ to the field F is called* **linear functional** *on V.*

(7.3.2) Example: *Let $V = \mathbb{R}^2$ be vector space over a field $\mathbb{R}$. Then a mapping $f : V \to \mathbb{R}$ defined by $f(x, y) = x - y$, is a linear functional on V because f is a linear transformation.*

(7.4) Dual spaces

(7.4.1) Definition: *Let V be a vector space over a field F. Then the vecor space $L(V,\, F)$ of all linear functionals on V is called* **dual space of V** *and is denoted by V^* i.e.*

$$V^* = \{f | f : V \to F \text{ is linear functional } \}.$$

(7.4.2) Example : *Let $V = \mathbb{R}^2$ be a vector space over $\mathbb{R}$. Then show that its dual space is given by $V^* =< (f_1,\, f_2) | f_1(x,\, y) = x$ and $f_2(x,\, y) = y >$.*

Solution *Let $B = \{(1,\, 0),\, (0,\, 1)\}$ be a basis of V. Define the rules $f_1 : V \to \mathbb{R}$ and $f_2 : V \to \mathbb{R}$ defined by $f_1(x,\, y) = a_1x + b_1y$ and $f_2(x,\, y) = a_2x + b_2y$ respectively,*

where $f_i(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$

Now, we shall show that for each $i = 1,\, 2,\, f_i$ is linear functional on V. For this, $f_1(a(x,\, y) + b(x',\, y')) = f_1(ax + bx',\, ay + by')$

$= a_1(ax + bx') + b_1(ay + by')$

$= a(a_1x + b_1y) + b(a_1x' + b_1y')$

$= af_1(x,\, y) + bf_1(x,\, y)$.

Similayly we can see that $f_2(a(x,\, y)+b(x',\, y')) = af_2(x,\, y)+bf_2(x,\, y)$ *. Therefore, $f_1,\, f_2$ are linear functionals on V.*

We claim that $B^* = \{f_1,\, f_2\}$ is a basis of V^*. First we check that B^* is linearly independent, for this cosider $\alpha_1f_1 + \alpha_2f_2 = O,\, \alpha_i \in F,\, i = 1,\, 2$

$\Rightarrow (\alpha_1f_1 + \alpha_2f_2)(e_i) = O(e_i)$

$\Rightarrow \alpha_1f_1(e_1) + \alpha_2f_2(e_1) = O(e_1)$

$\Rightarrow \alpha_1(1) + \alpha_2(0) = 0$

$\Rightarrow \alpha_1 = 0$

Similarily $\alpha_1f_1(e_2) + \alpha_2f_2(e_2) = O(e_2)$

$\Rightarrow \alpha_1(0) + \alpha_2(1) = 0$

$\Rightarrow \alpha_2 = 0$ *Therefore,* $\{f_1, f_2\}$ *is linearly independent. This implies that* B^* *is linearly independent.*

Now we shall show that B^* *spans* V^**. For this, let* $f \in V^*$ *be any element and* $f(x_i) = a_i$ *for* $1 \leq i \leq 2$.

Then

$$f(x) = f(\sum_{i=1}^{2} \alpha_i e_i) = \sum_{i=1}^{2} \alpha_i f(e_i) = \sum_{i=1}^{2} \alpha_i a_i.$$

Now

$$f_i(x) = f_i(\sum_{i=1}^{2} \alpha_j e_j) = \sum_{i=1}^{2} \alpha_j f_i(e_j) = \alpha_i.$$

Therefore

$$f(x) = \sum_{i=1}^{2} a_i f_i(x).$$

This implies that

$$f = \sum_{i=1}^{2} a_i f_i$$

Thus $V^* = L(B^*)$.

(7.4.3) Example *Let* $V = \{a + bx + cx^2 | a, b, c \in \mathbb{R}\}$ *be a vector space of polynomials of degree* ≤ 2. *Let* $f_1(p(t)) = \int_0^1 p(t)\, dt$; $f_2(p(t)) = p'(1)$ *and* $f_3(p(t)) = p(0)$ *for all* $p(t) \in V$. *Find the basis* $\{p_1, p_2, p_3\}$ *which is dual to* $\{f_1, f_2, f_3\}$.

Solution *Let* $p_1(t) = a_1 + b_1 t + c_1 t^2$, $p_2(t) = a_2 + b_2 t + c_2 t^2$, $p_3(t) = a_3 + b_3 t + c_3 t^2$ *be any element of* V*, where* a_i, b_i, c_i *are real numbers for* $i = 1, 2, 3$.

Now, $f_1(p_1(t)) = \int_0^1 (a_1 + b_1 t + c_1 t^2)\, dt = a_1 + \frac{b_1}{2} + \frac{c_1}{3}$

$f_2(p_1(t)) = \frac{d}{dt}(a_1 + b_1 t + c_1 t^2)|_{t=1} = b_1 + 2c_1$

$f_3(p_1(t)) = p_1(0) = a_1$.

Now by the definition of basis of V^* *as done in above example, we have* $f_1(p_1) = 1$, $f_2(p_1) = 0$, $f_3(p_1) = 0$

$\Rightarrow 1 = a_1 + \frac{b_1}{2} + \frac{c_1}{3}$, $b_1 + 2c_1 = 0$, $a_1 = 0$

$\Rightarrow a_1 = 0,\ b_1 = 3,\ c_1 = \frac{-3}{2}$

$\Rightarrow p_1(t) = 3t - \frac{3}{2}t^2 = 3t - \frac{3t^2}{2}.$

Similarly $f_1(p_2(t)) = \int_0^1 (a_2 + b_2 t + c_2 t^2)\, dt = a_2 + \frac{b_2}{2} + \frac{c_2}{3}$

$f_2(p_2(t)) = \frac{d}{dt}(a_2 + b_2 t + c_2 t^2)|_{t=1} = b_2 + 2c_2$

$f_3(p_2(t)) = p_2(0) = a_2$

Now by the definition of basis of V^* *, we have* $f_1(p_2) = 0,\ f_2(p_2) = 1,\ f_3(p_2) = 0$

$\Rightarrow a_2 + \frac{b_2}{2} + \frac{c_2}{3} = 0,\ b_2 + 2c_2 = 1,\ a_2 = 0$

$\Rightarrow a_2 = 0,\ b_2 = \frac{-1}{2},\ c_2 = \frac{3}{4}$

$\Rightarrow p_2(t) = \frac{-1}{2}t + \frac{3}{4}t^2 = \frac{-t}{2} + \frac{3t^2}{4}.$

Also, $f_1(p_3(t)) = \int_0^1 (a_3 + b_3 t + c_3 t^2)\, dt = a_3 + \frac{b_3}{2} + \frac{c_3}{3}$

$f_2(p_3(t)) = \frac{d}{dt}(a_3 + b_3 t + c_3 t^2)|_{t=1} = b_3 + 2c_3$

$f_3(p_3(t)) = p_3(0) = a_3$

Now by the definition of basis of V^* *, we have* $f_1(p_3) = 0,\ f_2(p_3) = 1,\ f_3(p_3) = 0$

$\Rightarrow a_3 + \frac{b_3}{2} + \frac{c_3}{3} = 0$

$b_3 + 2c_3 = 0$

$a_3 = 1$

$\Rightarrow a_3 = 1,\ b_3 = -3,\ c_3 = \frac{3}{2}$

$\Rightarrow p_3(t) = 1 - 3t + \frac{3}{2}t^2.$

Thus the required basis for V^* *is* $\{3t - \frac{3t^2}{2},\ \frac{-t}{2} + \frac{3t^2}{4},\ 1 - 3t + \frac{3t^2}{2}\}.$

Let Us Sum Up (7.5) *In this lesson we have defined linear functional and dual space of a finite dimensional vector space, then illustrated these concepts with examples. As we have learnt that knowing the basis of vector space is enough to know the vector space. Therefore in order to compute the dual space of given vector space it is enough to compute the basis of it which we have done* *in few examples.*

(7.6) **Lesson End Excercise**

1. *Find the dual space of the vector space* $\mathbb{R}^3$ *with respect to the standrad basis.*

2. *Let* $V = \mathbb{R}^3$ *be a vector space with basis*

$$\{(1,\,-1,\,3),\,(0,\,1,\,-1),\,(0,\,3,\,-2)\}.$$

Find basis of dual space V^*.

3. *Let* $B = \{(-1,\,1,\,1),\,(1,\,-1,\,-1),\,(1,\,1,\,-1)\}$ *be a basis of* $\mathbb{R}^3$. *Find basis of dual space* V^*.

(7.7) Suggested Readings *:(i) N.S. Gopalakrishnan, University Algebra, New Age International (P) Limited, Publishers.*

(ii) Kenneth Hoffman, Ray Kunze, Linear Algebra,Prentice Hall India.

Lesson-VIII

Dimension of Dual space of finite dimensional vector space

(8.1) Introduction: *With the given vector space we can construct its dual space. It is interesting to know the dimension of dual space that turns out to be same as that of vector space in case of finite dimensional vector spaces.*

(8.2) Objectives: *In this lesson, the students will learn to compute explicitly a basis for dual vector space knowing the basis of given vector space.*

(8.3) Dimension of dual space

(8.3.1) Theorem: *Let V be a finite dimensional vector space over F. Then*

$$\dim V = \dim V^*.$$

Proof *Let $B = \{x_1, x_2, \ldots, x_n\}$ be a basis of V. Now, for each i, $1 \leq i \leq n$,* *define mapping* $f_i : V \to F$ *by*

$$f_i(\sum_{i=1}^{n} \alpha_i x_i) = \alpha_i.$$

Then f_i *is a linear functional on* V, *for each* i *and* $B^* = \{f_1,\, f_2,\, \ldots,\, f_n\}$ *is a basis of* V^*.

For this, let $x, y \in V$ *and* $\alpha, \beta \in F$, *then for each* i, $f_i(\alpha x + \beta y) = f_i(\alpha \sum_{j=1}^{n} \alpha_j x_j + \beta \sum_{j=1}^{n} \beta_j x_j)$

$$\begin{aligned} &= f_i(\textstyle\sum_{j=1}^{n}(\alpha\alpha_j + \beta\beta_j)x_j) \\ &= \alpha\alpha_i + \beta\beta_i \\ &= \alpha f_i(\textstyle\sum_{j=1}^{n} \alpha_j x_j) + \beta f_i(\textstyle\sum_{j=1}^{n} \beta_j x_j) \\ &= \alpha f_i(x) + \beta f_i(y). \end{aligned}$$

Therefore, for each i, f_i *is a linear functional.*

Further, for each i, j *such that* $1 \le i,\, j \le n$ *we have*

$$f_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

We claim that B^* *is a basis of* V^*. *First we check that* B^* *is linearly independent, for this cosider*

$$\begin{aligned} \alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_n f_n &= O,\, \alpha_i \in F \\ \Rightarrow (\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_n f_n)(x_i) &= O(x_i) \\ \Rightarrow \alpha_1 f_1(x_i) + \alpha_2 f_2(x_i) + \ldots + \alpha_n f_n(x_i) &= O \\ \Rightarrow \alpha_1(0) + \alpha_2(0) + \ldots + \alpha_i(1) + \ldots + \alpha_n(0) &= 0 \\ \Rightarrow \alpha_i &= 0 \,\forall\, 1 \le i \le n. \end{aligned}$$

Therefore, $\{f_1,\, f_2,\, \ldots,\, f_n\}$ *is linearly independent. This implies that* B^* *is linearly independent.*

Now we shall show that B^* spans V^*. For this, let $f \in V^*$ be any element and $f(x_i) = a_i$ for $1 \leq i \leq n$. Then

$$
\begin{aligned}
f(x) &= f(\sum_{i=1}^{n} \alpha_i x_i) \\
&= \sum_{i=1}^{n} \alpha_i f(x_i) \\
&= \sum_{i=1}^{n} \alpha_i a_i \\
\textit{Now} \quad f_i(x) &= f_i(\sum_{i=1}^{n} \alpha_j x_j) \\
&= \sum_{i=1}^{n} \alpha_j f_i(x_j) \\
&= \alpha_i \\
\Rightarrow f(x) &= \sum_{i=1}^{n} a_i f_i(x) \\
\Rightarrow f &= \sum_{i=1}^{n} a_i f_i
\end{aligned}
$$

Hence $f \in L(B^*)$. This shows that $L(B^*) = V^*$ and B^* is a basis of V^*. Therefore

$$\dim V = \dim V^*.$$

(8.4) Dual Basis

(8.4.1) Definition *Let $B = \{x_1, x_2, \ldots, x_n\}$ be a basis of a finite dimensional vector space V. Then for each $1 \leq i \leq n$ the set of linear functionals on V defined as*

$$f_i(x_j) = \begin{cases} 1 & \textit{if } i = j \\ 0 & \textit{if } i \neq j \end{cases}$$

forms a basis of V^* *and is known as* **dual basis of** B. *It is denoted by* $B^* = \{f_1, f_2, \ldots, f_n\}$.

(8.4.2) Theorem *Let* V *be a finite dimensional vecor space over the field* F *and* $0 \neq x \in V$. *Then there exists a linear functional* f *on* V *such that* $f(x) \neq 0$.

Proof *Since* $x \neq 0$ *in* V. *Therefore,* $\{x\}$ *is a linearly independent suset of* V *ans so it can be extended to form a basis of* V. *Thus, there exists a basis* $B = \{x_1, x_2, \ldots, x_n\}$ *of* V *such that* $x = x_1$. *If* $B' = \{f_1, f_2, \ldots, f_n\}$ *is the dual basis, then* $f_1(x) = 1$ *and* $f_1(x_i) = 0$. *This implies that*$f_1(x) = f_1(x_1) = 1 \neq 0$. *Thus there exists* $f \in V^*$ *such that* $f(x) \neq 0$.

(8.4.3) Corollary *Let* V *be a finite dimensional vector space over the field* F *and* $f(x) = 0, \forall\ f\ \in V^*$. *Then* $x = 0$.

Proof *Suppose that* $x \neq 0$. *Then by the theorem, there exists a linear functional* f *on* V *such that* $f(x) \neq 0$ *which is a contradiction to the fact that* $f(x) = 0, \forall\ f\ \in V^*$.

(8.4.4) Corollary *Let* V *be a finite dimensional vector space over the field* F *and* x, y *be two different vectors in* V. *Then there exists a linear functional* f *on* V *such that* $f(x) \neq f(y)$.

Proof *Here we have* $x \neq y \Rightarrow x - y \neq 0$. *Then by the theorem* (8.2.4), *there exists a linear functional* f *on* V *such that*

$f(x - y) \neq 0 \Rightarrow f(x) - f(y) \neq 0$

$\Rightarrow f(x) \neq f(y)$.

(8.5) Examples 1. *Find dual basis for the basis* $\{(1, 0), (0, 1)\}$ *of* $\mathbb{R}^2$.

Solution *Let* $B = \{(1, 0), (0, 1)\} = \{e_1, e_2\}$ *be a basis of of* $\mathbb{R}^2$ *and*

$B^* = \{f_1,\, f_2\}$ *be its dual basis such that*

$$\begin{aligned} f_1(x,\, y) &= a_1x + b_1y \\ f_2(x,\, y) &= a_2x + b_2y \quad \textit{where } f_i(e_j) = \begin{cases} 1 & \textit{if } i = j \\ 0 & \textit{if } i \neq j \end{cases} \\ \textit{Now} \quad f_1(1,\, 0) &= a_1 \Rightarrow a_1 = 1 \\ f_1(0,\, 1) &= b_1 \Rightarrow b_1 = 0 \\ f_2(1,\, 0) &= a_2 \Rightarrow a_2 = 0 \\ f_2(0,\, 1) &= b_2 \Rightarrow b_2 = 1 \end{aligned}$$

Therefore $f_1(x,\, y) = x$ *and* $f_2(x,\, y) = y$ *so that* $B^* = \{x,\, y\}$ *is the required dual basis.*

2. *Let* $B = \{(1,\, -1,\, 3),\, (0,\, 1,\, -1),\, (0,\, 3,\, -2)\}$ *be a basis of* $\mathbb{R}^3$. *Find its dual basis.*

Solution *Let* $B = \{e_1,\, e_2,\, e_3\} = \{(1,\, -1,\, 3),\, (0,\, 1,\, -1),\, (0,\, 3,\, -2)\}$ *be the given basis of* $\mathbb{R}^3$. *Since the dimension of the vector space is same as the dimension of its dual space. Therefore the dual basis of* B *contains* 3 *elements. Let* $B^* = \{f_1,\, f_2,\, f_3\}$ *be the dual basis of* B *such that*

$$\begin{aligned} f_1(x) &= a_1x + a_2y + a_3z \\ f_2(x) &= b_1x + b_2y + b_3z \\ f_3(x) &= c_1x + c_2y + c_3z \ \textit{where } f_i(e_j) = \begin{cases} 1 & \textit{if } i = j \\ 0 & \textit{if } i \neq j \end{cases} \end{aligned}$$

$\Rightarrow f_1(e_1) = 1,\ f_1(e_2) = 0,\ f_1(e_3) = 0$
$\Rightarrow f_2(e_1) = 0,\ f_2(e_2) = 1,\ f_2(e_3) = 0$

$\Rightarrow f_3(e_1) = 0,\ f_3(e_2) = 0,\ f_3(e_3) = 1$

$$\begin{aligned}
f_1(e_1) &= a_1 - a_2 + 3a_3 \Rightarrow 1 = a_1 - a_2 + 3a_3 \\
f_1(e_2) &= a_2 - a_3 \Rightarrow 0 = a_2 - a_3 \\
f_1(e_3) &= 3a_2 - 2a_3 \Rightarrow 0 = 3a_2 - 2a_3
\end{aligned}$$

Solving these equations, we get $a_1 = 1,\ a_2 = 0,\ a_3 = 0$. *Therefore* $f_(x,\ y,\ z) = x$.

Similarily,

$$\begin{aligned}
f_2(e_1) &= b_1 - b_2 + 3b_3 \Rightarrow 0 = b_1 - b_2 + 3b_3 \\
f_2(e_2) &= b_2 - b_3 \Rightarrow 1 = b_2 - b_3 \\
f_2(e_3) &= 3b_2 - 2b_3 \Rightarrow 0 = 3b_2 - 2b_3
\end{aligned}$$

Solving these equations, we get $b_1 = 7,\ b_2 = -2,\ b_3 = -3$. *Therefore* $f_2(x,\ y,\ z) = 7x - 2y - 3z$. *Also,*

$$\begin{aligned}
f_3(e_1) &= c_1 - c_2 + 3c_3 \Rightarrow 0 = c_1 - c_2 + 3c_3 \\
f_3(e_2) &= c_2 - c_3 \Rightarrow 0 = c_2 - c_3 \\
f_3(e_3) &= 3c_2 - 2c_3 \Rightarrow 0 = 3c_2 - 2c_3
\end{aligned}$$

Solving these equations, we get $c_1 = -2,\ c_2 = 1,\ c_3 = 1$.

Therefore $f_3(x,\ y,\ z) = -2x + y + z$ *and the required dual basis is* $B^* = \{x,\ 7x - 2y - 3z,\ -2x + y + z\}$.

3. *Let* V *be the vector space of all polynomials in* t *over* $\mathbb{R}$ *of degree* ≤ 2. *Let* $t_1,\ t_2,\ t_3$ *be distinct real numbers and* $T_i : V \to F$ *be linear functions defined as* $T_i(f(x)) = f(t_i)$ *for* $i = 1,\ 2,\ 3$.

Prove that (i) $\{T_1,\ T_2,\ T_3\}$ *is a basis of* V^*.

(ii) Find a basis of V *such that* $\{T_1,\ T_2,\ T_3\}$ *is its dual basis.*

Solution*(i) Consider a linear combination* $\alpha T_1 + \beta T_2 + \gamma T_3 = O$

$(\alpha T_1 + \beta T_2 + \gamma T_3)(i) = O(e_i)$, $\forall\ i = 1, 2, 3.$

$\alpha T_1(1) + \beta T_2(1) + \gamma T_3(1) = 0$

$\Rightarrow \alpha + \beta + \gamma = 0.........(1)$

$\alpha T_1(t) + \beta T_2(t) + \gamma T_3(t) = 0$

$\Rightarrow \alpha t_1 + \beta t_2 + \gamma t_3 = 0..........(2)$

$\alpha T_1(t^2) + \beta T_2(t^2) + \gamma T_3(t^2) = 0$

$\Rightarrow \alpha {t_1}^2 + \beta {t_2}^2 + \gamma {t_3}^2 = 0..........(3)$

From (1), (2), *and* (3), *we get* $\begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ {t_1}^2 & {t_2}^2 & {t_3}^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Now $\begin{vmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ {t_1}^2 & {t_2}^2 & {t_3}^2 \end{vmatrix} = (t_2 - t_1)(t_3 - t_2)(t_3 - t_1) \neq 0$

$\Rightarrow \alpha = \beta = \gamma = 0.$ *Hence* $\{T_1, T_2, T_3\}$ *is linearly independent. Since* $\dim V^* = 3$ *so* $\{T_1, T_2, T_3\}$ *is a basis of* V^*.

(ii) Let $B = \{p_(x), p_2(x), p_3(x)\}$ *be a basis of* V *for which* $B^* = \{T_1, T_2, T_3\}$ *is its dual basis.*

Then $T_1(p_1(x)) = 1$, $T_2(p_1(x)) = 0$, $T_3(p_1(x)) = 0$.

$\Rightarrow p_1(t_1) = 1,\ p_1(t_2) = 0,\ p_1(t_3) = 0$

$\Rightarrow x - t_2,\ x - t_3$ *are the factors of* $p_1(x)$. *Thus,* $p_1(x) = \frac{(x-t_2)(x-t_3)}{(t_1-t_2)(t_1-t_3)}$.

Similarly, $T_1(p_2(x)) = 0$, $T_2(p_2(x)) = 1$, $T_3(p_2(x)) = 0$.

$\Rightarrow p_2(t_1) = 0,\ p_2(t_2) = 1,\ p_2(t_3) = 0$

$\Rightarrow x - t_1,\ x - t_3$ *are the factors of* $p_2(x)$. *Therefore,* $p_2(x) = \frac{(x-t_1)(x-t_3)}{(t_2-t_1)(t_2-t_3)}$.

Also $T_1(p_3(x)) = 0$, $T_2(p_3(x)) = 0$, $T_3(p_3(x)) = 1$.

$\Rightarrow p_3(t_1) = 0,\ p_3(t_2) = 0,\ p_3(t_3) = 1$

$\Rightarrow x - t_1,\ x - t_2$ *are the factors of* $p_3(x)$. *Therefore,* $p_3(x) = \frac{(x-t_1)(x-t_2)}{(t_3-t_1)(t_3-t_2)}$.

(8.6) Let Us Sum Up: *We have observed that the dimension of dual space of a finite dimensional vector space is the same. In this lesson we have defined dual basis for the given basis of a finite dimensional vector space and explicitly computed the dual basis for the given basis of a vector space.*

(8.7) **Lesson End Exercise**

1. Find the dual basis of $\{(1, 0, 0, 0), (0, 1, 0), (0, 0, 1)\}$ *of* $\mathbb{R}^3$.

2. Let $B = \{(-1, 1, 1), (1, -1, 1), (1, 1, -1)\}$ *be a basis of* $\mathbb{R}^3$. *Find dual basis for* B.

3. Find the dual basis of $\{(1, 1, 2), (0, 2, 1), (0, 0, 5)\}$.

4. Let V *be the vector space of all polynomials over the field* $\mathbb{R}$ *of degree* ≤ 1 *i.e.* $V = \{a + bx | a, b \in \mathbb{R}\}$. *Let* F_1, F_2 *be linear functionals on* V, *defined as*

$$f_1(p(x)) = \int_0^1 p(x)\, dx$$

and

$$f_2(p(x)) = \int_0^2 p(x)\, dx.$$

Find the basis of V *of which dual basis is* $\{f_1, f_2\}$.

Solution *Let* $\{v_1, v_2\} = \{a_1 + a_2x, b_1 + b_2x\}$ *be the required basis and* $B^* = \{f_1, f_2\}$ *be its dual basis. Then* $f_1(v_1) = 1$, $f_1(v_2) = 0$, $f_2(v_1) = 0$ *and* $f_2(v_2) = 1$.

Now

$$\begin{aligned} f_1(a_1 + a_2x) &= \int_0^1 (a_1 + a_2x)\, dx \Rightarrow 1 = a_1 + \frac{a_2}{2} \Rightarrow 2 = 2a_1 + a_2 \\ f_2(a_1 + a_2x) &= \int_0^2 (a_1 + a_2x)\, dx \Rightarrow 0 = 2a_1 + 2a_2 \end{aligned}$$

Solving these equations, we get $a_1 = 2$ *and* $a_2 = -2$. *Similarily*

$$\begin{aligned} f_1(b_1 + b_2x) &= \int_0^1 (b_1 + b_2x)\,dx \Rightarrow 0 = b_1 + \frac{b_2}{2} \Rightarrow 0 = 2b_1 + b_2 \\ f_2(b_1 + b_2x) &= \int_0^2 (b_1 + b_2x)\,dx \Rightarrow 1 = 2b_1 + 2b_2 \end{aligned}$$

Solving these equations, we get $b_1 = \frac{-1}{2}$ *and* $b_2 = 1$. *Therefore, the required basis of* V *is* $B = \{2 - 2x,\ \frac{-1}{2} + x\}$.

(8.8) **University Model Questions**

1. *Let* $V = \{a + bx + cx^2 | a,\ b,\ c \in \mathbb{R}\}$ *be a vector space of polynomials over* $\mathbb{R}$ *of degree* ≤ 2. *Let* $f_1,\ f_2,\ f_3$ *be linear functionals on* V, *defined as*

$$f_1(p(x)) = \int_0^1 p(x)\,dx,\ f_2(p(x)) = \int_0^2 p(x)\,dx,\ f_3(p(x)) = \int_0^{-1} p(x)\,dx.$$

Prove that $\{f_1,\ f_2,\ f_3\}$ *is a basis of* V^*.

2. *Let* $V(F)$ *be a vector space of dimension* n *and* $v_1,\ v_2$ *be two different vectors in* V, *then show that there exists* $f \in V^*$ *such that* $f(v_1) \neq f(v_2)$.

(8.9) **Suggested Readings** *:(i) N.S. Gopalakrishnan, University Algebra, New Age International (P) Limited, Publishers.*

(ii) Kenneth Hoffman, Ray Kunze, Linear Algebra,Prentice Hall India.

Unit-III

Lesson-IX **Linear Transformations on a vector space**

9.0 **Structure**

9.1 *Introduction*

9.2 *Objectives*

9.3 *Linear Transformations*

9.3.1 – 9.3.2 *Definitions*

9.3.3 – 9.3.12 *Theorems*

9.4 *Examples*

9.5 *Composition of linear Transformations*

9.5.1 *Definition*

9.5.2 *Example*

9.5.3 – 9.5.5 *Theorems*

9.6 *Linear Algebra*

9.6.1 *Definition*

9.6.2 *Theorem*

9.7 *Examples*

9.8 *Let Us Sum Up*

9.9 *Lesson end exercise*

9.10 *University Model Questions*

9.11 *Suggested Readings*

(9.1) Introduction: *Analogous to homomorphism in groups and rings, we can formulate the notion of homomorphism in vector spaces also. These are usually called* **linear transformations**. *In order to define a linear transformation between two vector spaces, it is necessary to assume both the vector* *spaces over the same field F.*

(9.2) Objectives: *(i) students will understand the concept of linear transformation with lot of examples.*

(ii) In this lesson the students will understand the algebra of linear transformations on a vector space.

(9.3) Linear transformations

(9.3.1) Definition: *Let V and V' be vector spaces over the field F. Then a mapping $T : V \to V'$ is said to be a linear transformation if*

(i) $T(x + y) = T(x) + T(y)$, x, $y \in V$

(ii)$T(\alpha x) = \alpha T(x)$, $\alpha \in F$ and $x \in V$.

(9.3.2) Definition : *Let V be a vector space over the field F. Then the linear transformation $T : V \to V$ is called* **linear operator** *on V.*

(9.3.3) Theorem : *Let V and V' be two vector spaces over the field F. Then the mapping $T : V \to V'$ is a linear transformation if and only if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$, $\forall\ \alpha$, $\beta \in F$ and $x, y \in V$.*

Proof *Already proved in Lesson* 6 *Theorem (6.1.1).*

(9.3.4) Theorem: *Let V and V' be vector spaces over a field F. Then a mapping $O : V \to V'$ defined as $O(x) = 0$, $\forall\ x \in V$ is a linear transformation.*

Proof *Let x, $y \in V$ and $\alpha \in F$. Then $\alpha x + \beta y \in V$ and $O(\alpha x + \beta y) = 0 = \alpha 0 + \beta 0 = \alpha O(x) + \beta O(y)$. Hence O is a linear transformation and it is called as* **zero transformation**.

(9.3.5) Theorem: *Let V be a vector space over a field F. Then the mapping $I : V \to V$ defined by $I(x) = x$, $\forall\ x \in V$ is a linear transformation (or Linear operator) on V.*

Solution *Let $x, y \in V$ and $\alpha, \beta \in F$. Then $\alpha x + \beta y \in V$. Now $T(\alpha x + \beta y) = \alpha x + \beta y = \alpha T(x) + \beta T(y)$.*

Therefore I is a linear operator called **identity operator** *on V.*

(9.3.6) Theorem *: Let V, V' be vector spaces over a field F and $T : V \to V'$ be a linear transformation. Then the mapping $-T : V \to V'$ defined by $(-T)(x) = -[T(x)]\,\forall\, x \in V$ is a linear transformation.*

Proof *We have $T : V \to V'$ is a linear transformation, so $T(x) \in V'$ for $x \in V \Rightarrow -T(x) \in V'$.*

Let x, $y \in V$ and α, $\beta \in F$.

Then $(-T)(\alpha x + \beta y) = -[T(\alpha x + \beta y)]$

$= -[\alpha T(x) + \beta T(x)]$

$= -\alpha T(x) - \beta T(y)$

$= \alpha(-T(x)) + \beta(-T(y))$

$= \alpha[(-T)(x)] + \beta[(-T)(y)]$

$\Rightarrow -T$ *is a linear transformation.*

(9.3.7) Theorem*: Let $T : V(F) \to V'(F)$ be a linear transformation. Then*

(i) $T(0) = 0'$ *(ii)* $T(-x) = -T(x)$

(iii) $T(x - y) = T(x) - T(y), \forall x, y \in V$ *(iv)* $T(mx) = mT(x), \forall\, m \in \mathbb{Z}$.

Proof *(i), (ii), (iii) already proved in Theorem* 6.2.1.

(iv) We shall prove this by induction on m.

Case I *When $m > 0$, For this, let $m = 1$, then $T(1x) = T(x) = 1T(x)$. So the result is true for $m = 1$.*

Now assume the result for $m = p$, p is a positive integer.

i.e. $T(px) = pT(x)$

Now $T((p + 1)x) = T(px + x)$

$= T(px) + T(x)$

$= pT(x) + T(x)$

$= (p + 1)T(x)$.

Therefore, the result is true for $m = (p+1)$. *Hence by induction,* $T(mx) = mT(x), \forall\ m \in\ \mathbb{N}$.

Case II *When* $m = 0$, *then* $T(0x) = T(0) = 0' = 0'T(x)$.

Therefore the result is true for $m = 0$.

Case III *When* $m < 0$, *let* $m = -p$, *where* p *is a positive integer.*

$$
\begin{aligned}
\text{Therefore } T(mx) &= T(-px) \\
&= T(p(-x)) \\
&= pT(-x) \\
&= p(-T(x)) \\
&= (-p)T(x) \\
&= mT(x)
\end{aligned}
$$

$\Rightarrow T(mx) = mT(x)$.

Hence the result is true for all $m \in \mathbb{Z}$.

(9.3.8) Theorem : *Let* V *and* W *be vector spaces over the same field* F. *Let* $B = \{x_1, x_2, \ldots, x_n\}$ *be a basis of* V *and* $y_1, y_2, \ldots, y_n$ *be any elements of* W. *Then there exists a unique linear transformation* $T : V \to W$ *such that* $T(x_i) = y_i$, $1 \le i \le n$.

Proof *Let* $x \in V$ *be any element. Then*

$$x = \sum_{i=1}^{n} a_i x_i$$

is the unique linear combination of elements of basis B. *Define a rule* $T : V \to W$ *such that* $T(a_1x_1 + a_2x_2 + \ldots + a_nx_n) = a_1y_1 + a_2y_2 + \ldots + a_ny_n$.

Since $a_1, a_2, \ldots, a_n$ *are unique, so the rule* T *is a well defined mapping.*

Now, each $x_i \in V$ *can be expressed as a linear combination of vectors of basis* B *i.e* $x_i = 0x_1 + \ldots + 1x_i + 0x_{i+1} + \ldots + 0x_n$.

Therefore $T(x_i) = 0y_1 + \ldots 1y_i + \ldots + 0y_n$

$\Rightarrow T(x_i) = y_i$, *for* $i = 1, 2, \ldots, n$.

T is linear transformation: *Let $x,\, x' \in V$ and $\alpha,\, \beta \in F$.*

Then

$$x = \sum_{i=1}^{n} a_i x_i \;\; \textit{and} \;\; x' = \sum_{i=1}^{n} a_i{}' x_i \;\; \textit{for}\, a_i,\, a_i{}' \in F\, \forall\, i.$$

Now,

$$\begin{aligned}
T(\alpha x + \beta x') &= T\left(\alpha \sum_{i=1}^{n} a_i x_i + \beta \sum_{i=1}^{n} a_i{}' x_i\right) \\
&= T\left(\sum_{i=1}^{n} (\alpha a_i) x_i + \sum_{i=1}^{n} (\beta a_i{}') x_i\right) \\
&= T\left(\sum_{i=1}^{n} (\alpha a_i + \beta a_i{}') x_i\right) \\
&= \sum_{i=1}^{n} (\alpha a_i + \beta a_i{}') y_i \\
&= \sum_{i=1}^{n} (\alpha a_i) y_i + \sum_{i=1}^{n} (\beta a_i{}') y_i \\
&= \alpha \sum_{i=1}^{n} a_i y_i + \beta \sum_{i=1}^{n} a_i{}' y_i \\
&= \alpha T(\sum_{i=1}^{n} a_i x_i) + \beta T(\sum_{i=1}^{n} a_i{}' x_i) \\
&= \alpha T(x) + \beta T(x') \\
\Rightarrow T(\alpha x + \beta x') &= \alpha T(x) + \beta T(x').
\end{aligned}$$

Hence T is a linear transformation.

T is unique: *Let $S : V \to W$ be another linear transformation such that $S(x_i) = y_i,\, i = 1,\, 2,\, \ldots,\, n$.*

Let $x = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$ be an element of V.

$S(x) = S(a_1 x_1 + a_2 x_2 + \ldots + a_n x_n)$

$\quad = a_1 S(x_1) + a_2 S(x_2) + \ldots + a_n S(x_n)$

$\quad = a_1 y_1 + a_2 y_2 + \ldots + a_n y_n$

$= T(x)$

$\Rightarrow S(x) = T(x), \forall\, x \in V$

Thus T is a unique linear transformation.

(9.3.9) Theorem*: Let V and W be vector spaces over the same field F and $T : V \to W$ be a linear transformation. Prove that (i) if $x_1, x_2, \ldots, x_n$ are linearly dependent over F then $T(x_1), T(x_2), \ldots, T(x_n)$ are also linearly dependent.*

(ii) if $x_1, x_2, \ldots, x_n$ are such that $T(x_1), T(x_2), \ldots, T(x_n)$ are linearly independent, then $x_1, x_2, \ldots, x_n$ are also linearly independent.

Proof *(i) Since $x_1, x_2, \ldots, x_n$ are linearly dependent. Therefore, there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zero, such that $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0$*

$\Rightarrow T(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n) = T(0)$

$\Rightarrow \alpha_1 T(x_1) + \alpha_2 T(x_2) + \ldots + \alpha_n T(x_n) = T(0)$

$\Rightarrow \alpha_1 T(x_1) + \alpha_2 T(x_2) + \ldots + \alpha_n T(x_n) = 0'$

Hence $T(x_1), T(x_2), \ldots, T(x_n)$ are linearly dependent in W.

(ii) Cosider $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0$, for $\alpha_1, \ldots, \alpha_n \in F$. Then

$T(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n) = T(0)$

$\Rightarrow T(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n) = T(0)$

$\Rightarrow \alpha_1 T(x_1) + \alpha_2 T(x_2) + \ldots + \alpha_n T(x_n) = 0'$

$\Rightarrow \alpha_1 T(x_1) + \alpha_2 T(x_2) + \ldots + \alpha_n T(x_n) = 0'$

$\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$ *(because $T(x_1), T(x_2), \ldots, T(x_n)$ are L.I).*

Hence $x_1, x_2, \ldots, x_n$ are linearly independent.

(9.3.10) Theorem *: Let V and W be two vector spaces over a field F. Then the set $L(V, W)$ of all linear transformations of V in W forms a vector space over F under the operations $+$ and scalar multiplication defined as $(T_1 + T_2)(x) = T_1(x) + T_2(x), \forall\, x \in V, \quad T_1, T_2 \in L(V, W)$ and $(\alpha T)(x) =$*

$\alpha T(x)$, $\forall\ x \in V$ and $\alpha \in F$, $T \in L(V, W)$ respectively.

proof *Already done in the exercise (6.7) problem number 4 of Lesson-VI.*

(9.3.11) Lemma: *Let V and V' be finite dimensional vector spaces over a field F and $B = \{x_1, x_2, \ldots, x_n\}$ be a basis of V. Then for any mapping $f : B \to V'$ there exists a unique linear transformation $T : V \to V'$ such that $T(x_i) = f(x_i)$, $\forall\ x_i \in B$, $i = 1, 2, \ldots, n$.*

Proof *Let $x \in V$ be any element. Then*

$$x = \sum_{i=1}^{n} a_i x_i.$$

Define a rule $T : V \to V'$ by

$T(a_1x_1 + a_2x_2 + \ldots + a_nx_n) = a_1 f(x_1) + a_2 f(x_2) + \cdots + a_n f(x_n).$

Then clearly, T is a well defined mapping because of the uniqueness of a_i's in the representation of x.

Now we shall show that T is a linear transformation.

For this, let

$$x = \sum_{i=1}^{n} a_i x_i,\ y = \sum_{i=1}^{n} b_i x_i$$

and α, $\beta \in F$. Then

$$\begin{aligned}
T(\alpha x + \beta y) &= T\left(\alpha \sum_{i=1}^{n} a_i x_i + \beta \sum_{i=1}^{n} b_i x_i\right) \\
&= T\left(\sum_{i=1}^{n} (\alpha a_i) x_i + \sum_{i=1}^{n} (\beta b_i) x_i\right) \\
&= T\left(\sum_{i=1}^{n} (\alpha a_i + \beta b_i) x_i\right) \\
&= \sum_{i=1}^{n} (\alpha a_i + \beta b_i) f(x_i)
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow T(\alpha x+\beta y) &= \sum_{i=1}^{n}\alpha a_i f(x_i)+\sum_{i=1}^{n}\beta b_i f(x_i)\\
&= \alpha\sum_{i=1}^{n} a_i f(x_i)+\beta\sum_{i=1}^{n} b_i f(x_i)\\
&= \alpha T\left(\sum_{i=1}^{n} a_i x_i\right)+\beta T\left(\sum_{i=1}^{n} b_i x_i\right)\\
&= \alpha T(x)+\beta T(y)\\
\Rightarrow T(\alpha x+\beta y) &= \alpha T(x)+\beta T(y).
\end{aligned}
$$

Therefore, T is a linear transformation.

Also, $T(x_i) = T(0x_1+\ldots+1x_i+\ldots 0x_n) = 0f(x_1)+\ldots 1f(x_i)+\ldots+0f(x_i) = f(x_i)$

$\Rightarrow T(x_i) = f(x_i), \forall\ i.$

For the uniqueness,

let S be another linear transformation such that $S(x_i) = f(x_i), \forall\ i$.

Then $S(x) = S(a_1x_1 + a_2x_2 + \ldots + a_nx_n)$

$= a_1S(x_1) + a_2S(x_2) + \ldots + a_nS(x_n)$

$= a_1f(x_1) + a_2f(x_2) + \ldots + a_nf(x_n)$

$= T(a_1x_1 + a_2x_2 + \ldots + a_nx_n)$

$= T(x) \Rightarrow S(x) = T(x), \forall\ x.$ *Thus $T = S$.*

Hence T is the unique linear transformation.

(9.3.12) Theorem *Let V and W be finite dimensional vector spaces over a field F. Then the vector space $L(V, W)$ of all linear transformations of V in W is also finite dimensional and*

$$\dim(L(V, W)) = (\dim V)(\dim W).$$

Proof *Let $B_1 = \{x_1, x_2, \ldots, x_n\}$ and $B_2 = \{y_1, y_2, \ldots, y_m\}$ be bases of V*

and W *respectively. Then for* $1 \le i \le n$, $1 \le j \le m$, *define a mapping*

$$T_{ij}(x_p) = \begin{cases} y_j & \text{if } i = p \\ 0 & \text{if } i \neq p \end{cases}$$

by the Lemma (9.2.11), T_{ij} *is a linear transformation for each* i, j.

Claim: $B = \{T_{ij} : 1 \le i \le n,\ 1 \le j \le m\}$ *is a basis of* $L(V, W)$. *For this, we first show that* B *is linearly independent. Let* α_{ij} *be set of* $m \times n$ *scalars, where* $1 \le i \le n$ *and* $1 \le j \le m$ *such that*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij} T_{ij} = O$$

Now $x_p \in V$ *for each* $p = 1, 2, \ldots, n$ *and* $O(x_p) = 0$. *Therefore,*

$$\begin{aligned}
\left(\sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij} T_{ij}\right)(x_p) &= O(x_p) \\
\sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij} T_{ij}(x_p) &= 0 \in W \\
\sum_{j=1}^{m}\sum_{i=1}^{n} \alpha_{ij} T_{ij}(x_p) &= 0 \\
\sum_{j=1}^{m} \alpha_{pj} y_j &= 0 \\
\alpha_{p1} y_1 + \alpha_{p2} y_2 + \ldots + \alpha_{pm} y_m &= 0 \\
\Rightarrow \alpha_{p1} = \alpha_{p2} = \ldots = \alpha_{pm} = 0, \text{ where } 1 \le p \le n & \\
\alpha_{ij} &= 0, \forall\, i, j
\end{aligned}$$

Hence B *is linearly independent.*

Now we shall show that B *is spanning set for* $L(V, W)$. *For this, let* $T \in L(V, W)$ *be any element so that* $T(x_p) \in W$. *Then*

$$T(x_p) = \sum_{j=1}^{m} \beta_{pj} y_j.$$

Now cosider

$$\begin{aligned}
\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\beta_{ij}T_{ij}\right)(x_p) &= \sum_{i=1}^{n}\sum_{j=1}^{m}\beta_{ij}T_{ij}(x_p) \\
&= \sum_{j=1}^{m}\beta_{pj}y_j \\
&= T(x_p) \\
\Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{m}\beta_{ij}T_{ij}(x_p) &= T(x_p) \\
\Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{m}\beta_{ij}T_{ij} = T
\end{aligned}$$

Hence B is a basis of $L(V, W)$ and

$$\dim(L(V, W)) = mn = (\dim V)(\dim W).$$

(9.4) Examples

1. *Show that the following mappings are linear transformations:*
(i) $T : V_3(R) \to V_2(R)$ defined by $T(x, y, z) = (x - y + z, 2x)$
(ii) $T : V(R) \to V(R)$ defined by $T(x + \iota y) = x - \iota y$
where $V(R) = \{x + \iota y | x, y \in \mathbb{R}$ and $\iota = \sqrt{-1}\}$.

Solution *(i) Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$ be any elements of $V_3(R)$ and α, β be any real numbers.*

Then $\alpha u + \beta v = (\alpha x_1 + \beta x_2,\, \alpha y_1 + \beta y_2,\, \alpha z_1 + \beta z_2) \in V_3(R)$. *Now,*

$$\begin{aligned}
T(\alpha u + \beta v) &= T(\alpha x_1 + \beta x_2,\, \alpha y_1 + \beta y_2,\, \alpha z_1 + \beta z_2)\\
&= (\alpha x_1 + \beta x_2 - \alpha y_1 - \beta y_2 + \alpha z_1 + \beta z_2,\, 2\alpha x_1 + 2\beta x_2)\\
&= ((\alpha x_1 - \alpha y_1 + \alpha z_1) + (\beta x_2 - \beta y_2 + \beta z_2),\, 2\alpha x_1 + 2\beta x_2)\\
&= (\alpha x_1 - \alpha y_1 + \alpha z_1,\, 2\alpha x_1) + (\beta x_2 - \beta y_2 + \beta z_2,\, 2\beta x_2)\\
&= \alpha(x_1 - y_1 + z_1,\, 2x_1) + \beta(x_2 - y_2 + z_2,\, 2x_2)\\
&= \alpha T(x_1,\, y_1,\, z_1) + \beta T(x_2,\, y_2,\, z_2)\\
\Rightarrow T(\alpha u + \beta v) &= \alpha T(u) + \beta T(v)
\end{aligned}$$

Hence T *is a linear transformation.*

(ii) Let $u = x_1 + \iota y_1$ *and* $v = x_2 + \iota y_2$ *be any elements of* $V(R)$ *and* $\alpha,\, \beta$ *be any real numbers. Then*

$$\begin{aligned}
T(\alpha u + \beta v) &= T(\alpha(x_1 + \iota y_1) + \beta(x_2 + \iota y_2))\\
&= T((\alpha x_1 + \beta x_2) + \iota(\alpha y_1 + \beta y_2))\\
&= (\alpha x_1 + \beta x_2) - \iota(\alpha y_1 + \beta y_2)\\
&= \alpha(x_1 - \iota y_1) + \beta(x_2 - \iota y_2)\\
&= \alpha T(x_1 + \iota y_1) + \beta T(x_2 + \iota y_2)\\
\Rightarrow T(\alpha u + \beta v) &= \alpha T(u) + \beta T(v)
\end{aligned}$$

Hence T *is a linear transformation.*

2. *Show that the following mappings are not linear transformations:*

(i) $T : \mathbb{R}^3 \to \mathbb{R}^2$ *, defined by* $T(x,\, y,\, z) = (|y|,\, 0)$

(ii)$T : \mathbb{R}^2 \to \mathbb{R}^3$ *defined by* $T(x,\, y) = (x + 1,\, 2y,\, x + y)$

(iii) $T : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $T(x,\, y) = xy$.

Solution *(i) Let* $u = (x_1,\, y_1,\, z_1)$ *and* $v = (x_2,\, y_2,\, z_2)$ *be any elements of* $\mathbb{R}^3$.

Then $u + v = (x_1 + x_2,\, y_1 + y_2 z_1 + z_2) \in \mathbb{R}^3$.

Now $T(u + v) = T(x_1 + x_2,\, y_1 + y_2,\, z_1 + z_2) = (|y_1 + y_2|,\, 0)$ *and*

$T(u) + T(v) = T(x_1,\, y_1,\, z_1) + T(x_2,\, y_2,\, z_2)$

$= (|y_1|,\, 0) + (|y_2|,\, 0) = (|y_1| + |y_2|,\, 0)$ *Therefore, we have* $T(u + v) \neq T(u) + T(v)$.

(ii) Let $u = (x_1,\, y_1)$ *and* $v = (x_2,\, y_2)$ *be any elements of* $\mathbb{R}^2$.

Then $T(u+v) = T(x_1 + x_2,\, y_1 + y_2) = (x_1 + x_2 + 1,\, 2(y_1 + y_2),\, x_1 + x_2 + y_1 + y_2)......(i)$

and $T(u) + T(v) = T(x_1,\, y_1) + T(x_2,\, y_2)$

$= (x_1 + 1,\, 2y_1,\, x_1 + y_1) + (x_2 + 1,\, 2y_2,\, x_2 + y_2)$

$= (x_1 + x_2 + 1 + 2,\, 2(y_1 + y_2),\, x_1 + x_2 + y_1 + y_2).........(ii)$.

Now from (i) and (ii) we have $T(u + v) \neq T(u) + T(v)$.

Hence T *is not a linear transformation.*

(iii) Let $u = (x_1,\, y_1)$ *and* $v = (x_2,\, y_2)$ *be any elements of* $\mathbb{R}^2$.

Then $T(u + v) = T(x_1 + x_2,\, y_1 + y_2)$

$= (x_1 + x_2)(y_1 + y_2)$

$= x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2......(i)$

Similarily $T(u) + T(v) = T(x_1,\, y_1) + T(x_2,\, y_2)$

$= x_1y_1 + x_2y_2.......(ii)$

Then from (i) and (ii) we get $T(u + v) \neq T(u) + T(v)$.

Hence T *is not a linear transformation.*

3. *Let* V *and* V' *be vector spaces over the field* F. *Show that the mapping* $T : V \to V'$ *is a linear transformation if and only if* $T(\alpha x+y) = \alpha T(x)+T(y)$.

Solution *First, suppose that* $T : V \to V'$ *is a linear transformation. Then it is obvious that* $T(\alpha x + y) = T(\alpha x) + T(y) = \alpha T(x) + T(y)$, *forall* $x,\, y \in V$ *and* $\alpha \in F$.

Conversely, suppose that $T(\alpha x+y)=\alpha T(x)+T(y)$. *Now take* $\alpha=1$, *then we get* $T(1x+y)=1T(x)+T(y)=T(x)+T(y)$

$\Rightarrow T(x+y)=T(x)+T(y)$*.......(1).*

Also take $y=0\in V$. *Then* $T(\alpha x+0)=T(\alpha x)=\alpha T(x)$

$\Rightarrow T(\alpha x)=\alpha T(x)$*........(2).*

Therefore from (1) *and* (2) *we see that* T *is a linear transformation.*

(9.5) **Composition of linear transformations**

(9.5.1) Definition: *Let* U, V, W *be vector spaces over the field* F *and* $T: U\to V$, $S:V\to W$ *be linear transformations. Then the composite mapping* $ST:U\to W$ *is defined as*

$$SoT(x)=(ST)(x)=S(T(x)),\,\forall\, x\in U.$$

(9.5.2) Example: *Let* V *be vector space of polynomials over reals. Define linear operators* D *and* T *as*

$$D(f(t))=\frac{df(t)}{dt}\ \text{and}\ T(f(t))=\int_0^t f(t)\,dt.$$

Show that $DT=I$ *and* $TD\neq I$, *where* I *is the identity operator.*

Solution *Let* $f(t)=a_0+a_1t+a_2t^2+\ldots$, *where* $a_i's$ *are real numbers.*

Then

$$\begin{aligned}
(DT)(f(t)) &= D[T(f(t))]\\
&= D\left[\int_0^1 f(t)\,dt\right]\\
&= D\left[\int_0^1 (a_0+a_1t+a_2t^2+\ldots)\,dt\right]\\
&= D\left[\left(a_0t+a_1\frac{t^2}{2}+a_2\frac{t^3}{3}+\ldots\right)_0^t\right]\\
&= a_0+a_1t+a_2t^2+a_3t^3+\ldots\\
&= f(t)
\end{aligned}$$

Therefore $DT(f(t)) = I(f(t))$

$\Rightarrow DT = I$.

Now

$$\begin{aligned}(TD)(f(t)) &= T[D(f(t))] \\ &= T[a_1 + 2a_2t + 3a_3t^2 + \ldots] \\ &= \int_0^t (a_1 + 2a_2t + 3a_3t^2 + \ldots)\, dt \\ &= \left[a_1t + a_2t^2 + a_3t^3 + \ldots\right]_0^t \\ &= a_1t + a_2t^2 + a_3t^3 + \ldots \neq f(t)\end{aligned}$$

Therefore $(TD)(f(t)) \neq f(t)$

$\Rightarrow TD \neq I$. *Hence* $DT \neq TD$.

(9.5.3) Theorem *Let* U, V, W *be vector spaces over the same field* F *and* $T_1 : U \to V$ *and* $T_2 : V \to W$ *be linear transformations. Then* $T_2T_1 : U \to W$ *is a linear transformation.*

Proof *Since* $T_1 : U \to V$ *and* $T_2 : V \to W$ *are linear transformations so the composite mapping* $T_2T_1 : U \to W$ *is defined by* $(T_2T_1)(x) = T_2(T_1(x))$, $\forall\ x \in U$.

Let x, $y \in U$ *and* α, $\beta \in F$. *Then*

$$\begin{aligned}(T_2T_1)(\alpha x + \beta y) &= T_2[T_1(\alpha x + \beta y)] \\ &= T_2[\alpha T_1(x) + \beta T_1(y)] \\ &= T_2(\alpha T_1(x)) + T_2(\beta T_1(y)) \\ &= \alpha T_2(T_1(x)) + \beta T_2(T_1(y)) \\ &= \alpha(T_2T_1)(x) + \beta(T_2T_1)(y).\end{aligned}$$

Hence $T_2T_1 : U \to W$ *is a linear transformation.*

(9.5.4) Theorem *Let* U, V, W *be vector spaces over the same field* F *and*

$T_1 : U \to V$ and $T_2 : U \to V$ be linear transformations. Also let $S_1 : V \to W$ and $S_2 : V \to$ be linear transformations. Then

(i) $S_1(T_1 + T_2) = S_1T_1 + S_1T_2$

(ii) $(S_1 + S_2)T_1 = S_1T_1 + S_2T_1$

(iii) $\alpha(S_1T_1) = (\alpha S_1)T_1 = S_1(\alpha T_1)$ *for* $\alpha \in F$.

Proof*(i) For each $x \in U$, we have*

$$\begin{aligned}
[S_1(T_1 + T_2)](x) &= S_1[(T_1 + T_2)(x)] \\
&= S_1[T_1(x) + T_2(x)] \\
&= S_1(T_1(x)) + S_1(T_2(x)) \\
&= (S_1T_1)(x) + (S_1T_2)(x) \\
&= [S_1T_1 + S_1T_2](x)
\end{aligned}$$

Hence $S_1(T_1 + T_2) = S_1T_1 + S_1T_2$.

(ii) For each $x \in U$, we have

$$\begin{aligned}
[(S_1 + S_2)T_1](x) &= (S_1 + S_2)(T_1(x)) = S_1(T_1(x)) + S_2(T_1(x)) \\
&= (S_1T_1)(x) + (S_2T_1)(x) = (S_1T_1 + S_2T_1)(x)
\end{aligned}$$

Hence $(S_1 + S_2)T_1 = S_1T_1 + S_2T_1$.

(iii) For all $x \in U$, we have

$$\begin{aligned}
[\alpha(S_1T_1)](x) &= \alpha(S_1T_1)(x) = \alpha S_1(T_1(x)) \\
&= [(\alpha S_1)T_1](x) ....(1) \\
\textit{Also}[S_1(\alpha T_1)](x) &= S_1[\alpha T_1(x)] \\
&= \alpha S_1(T_1(x)) \\
&= [(\alpha S_1)T_1](x) ....(2)
\end{aligned}$$

From (1) *and* (2)*, we get $\alpha(S_1T_1) = (\alpha S_1)T_1 = S_1(\alpha T_1)$.*

(9.5.5) Theorem*: Let R, S, T be three linear operators on a vector space*

$V(F)$ and O and I be the zero and identity operators on V. Then $(i) RO = OR = O$ $(ii) RI = IR = R$ $(iii) R(S+T) = RS + ST$ $(iv)(R+S)T = RT + ST$ $(v) R(ST) = (RS)T$ $(vi) k(RS) = (kR)S = R(kS)$.

Proof *Let $x \in V$. Then $(RO)(x) = R[O(x)] = R(0) = 0 = O(x)$*

$\Rightarrow RO = O$.

Similarly $(OR)(x) = O(R(x)) = O(y) = 0 = O(x)$

$\Rightarrow OR = O$, *where* $R(x) = y$.

$(ii)(RI)(x) = R(I(x)) = R(x)$

$\Rightarrow RI = R$ *and* $(IR)(x) = I(R(x)) = I(y) = y = R(x)$

$\Rightarrow IR = R$.

$(iii)[R(S+T)](x) = R[(S+T)(x)] = R[S(x) + T(x)]$

$= R(S(x)) + R(T(x))$

$= (RS)(x) + (RT)(x)$

$= (RS + RT)(x)$

$\Rightarrow R(S+T) = RS + RT$.

$(iv)[R(ST)](x) = R[(ST)(x)] = R[S(T(x))]$

$= [RS][T(x)]$

$= [(RS)T](x)$

$\Rightarrow R(ST) = (RS)T$.

$(v)[k(RS)](x) = k(RS)(x) = kR(S(x))$

$= (kR)(S(x)) = [(kR)S](x)$

$\Rightarrow k(RS) = (kR)S$ *and*

$$\begin{aligned} [R(kS)](x) &= R((kS)(x)) \\ &= R(kS(x)) \\ &= kR(S(x)) \\ &= k[R(S(x))] \\ &= [k(RS)](x) \end{aligned}$$

Hence $k(RS) = (kR)S = R(kS)$.

(9.6) Linear Algebra

(9.6.1) Definition: *Let V be a vector space over the field F. Then $V(F)$ is said to be an algebra over F if the following properties under the binary operation mutiplication are satisfied:*

(i) For all $x, y, z \in V$, $(xy)z = x(yz)$

(ii) For all $x, y, z \in V$, $x(y+z) = xy + xz$ *and* $(x+y)z = xz + yz$

(iii) For all $x, y \in V$, $\alpha \in F$, $\alpha(xy) = (\alpha x)y = x(\alpha y)$.

Note:*(1) If $xy = yx, \forall\ x, y \in V$, then V is a* **commutative algebra**.*(2) If there exists $1 \in V$ such that $1x = x1 = x$ for $x \in V$, then $V(F)$ is called* **linear algebra with unity**.

(9.6.2) Theorem *Let V be a vector space over a field F. Then $L(V, V)$, the set of linear operators on V is an algebra with unity.*

Proof *We have already proved in Lesson-VI problem 4 of exercise (6.7) that $L(V, V)$ is a vector space and the rest properties of algebra follow from the theorems* $(9.5.4 - 9.5.5)$.

(9.7) Examples

1. *Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ be two linear transformations defined as*

$T_1(x,\, y,\, z) = (3x,\, y + z); T_2(x,\, y,\, z) = (2x - 3z,\, y)$. Compute $T_1 + T_2$, $4T_1 - 5T_2$, T_1T_2, T_2T_1 if exist.

Solution $(T_1 + T_2)(x,\, y,\, z) = T_1(x,\, y,\, z) + T_2(x,\, y,\, z)$

$= (3x,\, y + z) + (2x - 3z,\, y)$

$= (5x - 3z,\, 2y + z)$

$\Rightarrow (T_1 + T_2)(x,\, y,\, z) = (5x - 3z,\, 2y + z)$.

Now, $(4T_1 - 5T_2)(x,\, y,\, z) = 4T_1(x,\, y,\, z) - 5T_2(x,\, y,\, z)$

$= 4(3x,\, y + z) - 5(2x - 3z,\, y)$

$= (12x - 10x + 15z,\, 4y + 4z - 5y)$

$= (2x + 15z,\, 4z - y)$

$\Rightarrow (4T_1 - 5T_2)(x,\, y,\, z) = (2x + 15z,\, 4z - y)$.

Here T_1T_2, T_2T_1 can not be defined.

2. *Let T_1 and T_2 be linear operators on $\mathbb{R}^2$ defined by $T_1(x,\, y) = (y,\, x)$ and $T_2(x,\, y) = (x,\, 0)$.*

Compute $T_1 + T_2$, T_2T_1, T_1T_2, ${T_1}^2$, ${T_2}^2$.

Solution $(T_1 + T_2)(x,\, y) = T_1(x,\, y) + T_2(x,\, y)$

$$= (y,\, x) + (x,\, 0) = (y + x,\, x)$$

$\Rightarrow (T_1 + T_2)(x,\, y) = (y + x,\, x)$.

$(T_1T_2)(x,\, y) = T_1(T_2(x,\, y)) = T_1(x,\, 0) = (x,\, x)$

$\Rightarrow T_1T_2(x,\, y) = (x,\, x)$.

Similarly, $(T_2T_1)(x,\, y) = T_2(T_1(x,\, y)) = T_2(y,\, x) = (y,\, 0)$

$\Rightarrow (T_2T_1)(x,\, y) = (y,\, 0)$.

Now, ${T_1}^2(x,\, y) = T_1(T_1(x,\, y)) = T_1(y,\, x) = (x,\, y)$

$\Rightarrow {T_1}^2 = I$

Also, ${T_2}^2(x,\, y) = T_2(T_2(x,\, y)) = T_2(x,\, 0) = (x,\, 0)$

$\Rightarrow {T_2}^2(x,\, y) = (x,\, 0)$.

3. *Find a linear transformation which transforms*

$$(3,\,-1,\,-2),\,(1,\,1,\,0),\,(-2,\,0,\,2) \in \mathbb{R}^3$$

to twice the elementary vectors i.e. $2e_1,\,2e_2,\,2e_3$ *in* $\mathbb{R}^3$*, where* $e_1,\,e_2,\,e_3$ *are elementary vectors.*

Solution *Let* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *be a linear transformation such that* $T(3,\,-1,\,-2) = 2e_1,\,T(1,\,1,\,0) = 2e_2,\,T(-2,\,0,\,2) = 2e_3.$

First, we show that $B = \{(3,\,-1,\,-2),\,(1,\,1,\,0),\,(-2,\,0,\,2)\}$ *is a basis of* V*. For this, it is enough to show that* B *is linearly independent. Consider*

$a(3,\,-1,\,-2) + b(1,\,1,\,0) + c(-2,\,0,\,2) = (0,\,0,\,0)$

$\Rightarrow (3a + b - 2c,\,-a + b,\,-2a + 2c) = (0,\,0,\,0)$

$\Rightarrow 3a + b - 2c = 0$

$-\,a + b = 0$

$-\,2a + 2c = 0$

$$\begin{bmatrix} 3 & 1 & -2 \\ -1 & 1 & 0 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{vmatrix} 3 & 1 & -2 \\ -1 & 1 & 0 \\ -2 & 0 & 2 \end{vmatrix} = 4 \neq 0.$ *Hence* $a = b = c = 0 \Rightarrow B$ *is linearly independent and thus a basis of* $\mathbb{R}^3$.

Now, let $(x,\,y,\,z)$ *be any element of* $\mathbb{R}^3$*. Then* $(x,\,y,\,z) = \alpha(3,\,-1,\,-2) + \beta(1,\,1,\,0) + \gamma(-2,\,0,\,2)$

$\Rightarrow 3\alpha + \beta - 2\gamma = x$

$-\,\alpha + \beta = y$

$-\,2\alpha + 2\gamma = z$*. Solving these equations we get* $\alpha = \frac{1}{2}(x - y + z),\,\beta = y + \alpha$

$\Rightarrow \beta = \frac{1}{2}(x + y + z)$ *and* $\gamma = \frac{1}{2}(x - y + 2z)$

The required linear transformation is given by $T(x, y, z) = T\left(\frac{1}{2}(x-y+z)(3, -1, -2) + \frac{1}{2}(x+y+z)(1, 1, 0) + \frac{1}{2}(x-y+2z)(-2, 0, 2)\right)$
$= \frac{1}{2}(x-y+z)2e_1 + \frac{1}{2}(x+y+z)2e_2 + \frac{1}{2}(x-y+2z)2e_3$
$= (x-y+z, x+y+z, x-y+2z)$
$\Rightarrow T(x, y, z) = (x-y+z, x+y+z, x-y+2z)$.

4. *Show that the following mappings are linear transformations:*
(i)$T : \mathbb{R}^3 \to \mathbb{R}$ *defined by* $T(x, y, z) = x + 3y - 4z$
(ii)$T : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $T(x, y) = (0, -x)$.

Solution *(i) Let* $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$ *and* $\alpha, \beta \in F$. *Then* $\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) \in \mathbb{R}^3$.
Now $T(\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)) = T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$
$= \alpha x_1 + \beta x_2 + 3(\alpha y_1 + \beta y_2) - 4(\alpha z_1 + \beta z_2)$
$= \alpha(x_1 + 3y_1 - 4z_1) + \beta(x_2 + 3y_2 - 4z_2)$
$= \alpha T(x_1, y_1, z_1) + \beta T(x_2, y_2, z_2)$.
Hence T *is a linear transformation.*

(9.8) Let Us Sum Up: *Basically linear algebra began with the study of linear equations. In order to define linear algebra we have defined linear transformations and linear operators on vector space, their composition. Then we have illustrated them with various examples. At the end with the help of theorems we could able to define linear algebra i.e. algebra of linear operators. The set which is having both the structures vector space and ring.*

(9.9) **Lesson End Exercise**

1. *Let* $T : \mathbb{R} \to \mathbb{R}^2$ *defined by* $T(x) = (2x, 3x)$. *Show that* T *is a linear transformation.*
2. *Let* $V(\mathbb{R})$ *be a vector space of integrable functions on* $\mathbb{R}$. *Prove that* T :

$V \to \mathbb{R}$ defined by

$$T(f) = \int_c^d f(x)\, dx; c,\, d \in \mathbb{R}$$

is a linear functional.

3. *Show that the following mappings are not linear transformations:*

(i)$T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x,\, y) = |2x - 3y|$

(ii)$T : V(\mathbb{C}) \to V(R)$ defined by $T(x + \iota y) = (x^3 + y^3)^{\frac{1}{3}}$.

(iii)$T : \mathbb{C} \to \mathbb{C}$ defined by $T(x + \iota y) = x,\, \forall\, x,\, y \in \mathbb{R}$, where $\iota = \sqrt{-1}$.

4. *Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(1,\, 2) = (3,\, 4)$ and $T(0,\, 1) = (0,\, 0)$.*

Hint *(First, check that $\{(1,\, 2),\, (0,\, 1)\}$ forms a basis for $\mathbb{R}^2$ and find the mapping).*

5. *Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,\, 2) = (3,\, -1,\, 5)$ and $T(0,\, 1) = (2,\, 1,\, -1)$.*

6. *Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(2,\, 3) = (1,\, 2)$ and $T(3,\, 2) = (2,\, 3)$.*

7. *Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. Prove that $(T^2 - I)(T - 3I) = O$.*

8. *Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^3$ be two linear transformations defined by $T(x,\, y,\, z) = (x - 3y - 2z,\, y - 4z)$ and $S(x,\, y) = (2x,\, 4x - y,\, 2x + 3y)$. Find TS, ST. Is product commutative?*

9. *Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^3$ be two linear transformations defined by $T_1(x,\, y,\, z) = (3x,\, y + z),\, T_2(x,\, y,\, z) = (2x - 3z,\, y)$. Compute $T_1 + T_2,\, 5T_1,\, 4T_1 - 5T_2,\, T_1T_2$ and T_2T_1.*

10. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be LT defined by $T(x,\, y) = (x + y,\, 2x)$. Let $f(t) = t^2 - 2t + 3$. Find $f(T)(x,\, y)$.*

Hint. $$\begin{aligned} f(T)(x,\, y) &= (T^2 - 2T + 3I)(x,\, y) \\ &= T^2(x,\, y) - 2T(x,\, y) + 3I(x,\, y) \end{aligned}$$

$= T(T(x, y)) - 2(x + y, 2x) + 3(x, y))$

$= T(x + y, 2x) + (-2x - y, -2x) + (3x, 3y)$

$= (3x + y, 2x + 2y) + (-2x - y, -2x) + (3x, 3y)$

$= (4x, 5y) \Rightarrow f(T)(x, y) = (4x, 5y).$

(9.10) University Model Questions

1. *Define linear transformation. Show that the mapping* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by* $T(x, y, z) = (y, -x, -z)$ *is a linear transformation.*

2. *Define linear transformation. For a linear transformation* T *show that (i)* $T(0) = 0$ *and (ii)*$T(x - y) = T(x) - T(y)$.

3. *Let* $V(F)$ *be a vector space of all* $m \times n$ *matrices over a field* F *and let* P *and* Q *be two square matrices of orders* $m \times m$ *and* $n \times n$ *respectively. Show that the mapping* $T : V \to V$ *defined as* $T(A) = PAQ, \forall A \in V$ *is a linear transformation.*

4. *Define linear transformation. Show that the mapping* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by* $T(x, y, z) = (y, -x + 1, -z)$ *is not a linear transformation.*

(9.11) Suggested Readings:*(i) N.S. Gopalakrishnan, University Algebra, New Age International (P) Limited, Publishers.*

(ii) Kenneth Hoffman, Ray Kunze, Linear Algebra,Prentice Hall India.

Lesson-X Matrix representation of Linear Transformation

(10.1) Introduction: *In this lesson we establish relationship between linear transformations and matrices. Then we translate the properties of linear transformations to the corresponding properties of the matrices and vice-versa.*

(10.2) Objective: *Students will get the feeling about the operations on matrices with the help of the correspondence between linear transformations and matrices.*

(10.3) Matrix representation of a linear transformation

(10.3.1) Definition: *An $m \times n$ matrix over a field F is an array of an mn elements of F of the form*

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

(10.3.2) Theorem: *Let V be an n-dimensional vector space over the field*

F and W an m-dimensional vector space over the field F. Let $B_1 = \{x_1, x_2, \ldots, x_n\}$ and $B_2 = \{y_1, y_2, \ldots, y_m\}$ be bases for V and W respectively. Then for each linear transformation $T : V \to W$, there is an $m \times n$ matrix with entries in F and vice-a-versa.

Proof *First, we suppose that $T : V \to W$ is a linear transformation. Then*

$$\begin{aligned}
T(x_1) &= a_{11}y_1 + a_{21}y_2 + \ldots a_{m1}y_m \\
T(x_2) &= a_{12}y_1 + a_{22}y_2 + \ldots a_{m2}y_m \\
T(x_3) &= a_{13}y_1 + a_{23}y_2 + \ldots a_{m3}y_m \\
&\vdots \\
T(x_n) &= a_{1n}y_1 + a_{2n}y_2 + \ldots a_{mn}y_m
\end{aligned}$$

Therefore, the matrix corresponding to T is given by

$$m(T) = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}.$$

Conversely, suppose that $A = [a_{ij}]_{m\times n}$ be the given matrix and T be the linear transformation determined by the mn scalars a_{ij}.

Now, let $x \in V$. *Then* $x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$, $\alpha_i \in F \, \forall \, i$. *Therefore*

$$\begin{aligned} T(x) &= T\left(\sum_{i=1}^{n} \alpha_i x_i\right) \\ &= \sum_{i=1}^{n} \alpha_i T(x_i) \\ &= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} a_{ji} y_j \\ &= \sum_{j=1}^{m} (\sum_{i=1}^{n} a_{ji} \alpha_i) y_j \end{aligned}$$

$$\Rightarrow T\left(\sum_{i=1}^{n} \alpha_i x_i\right) = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ji}\alpha_i\right) y_j$$

(10.4) Examples

1. *Let* T *be a linear operator on* $\mathbb{R}^2$ *defined by*

$$T(x,\, y) = (4x - 2y,\, 2x + y).$$

(i) Find the matrix of T *relative to the basis* $B = \{(1,\, 1),\, (-1,\, 0)\}$.

(ii) Also, verify the linear transformation corresponding to the matrix $m(T)$.

Solution *(i) We have a linear operator* T *on* V *given by*

$$T(x,\, y) = (4x - 2y,\, 2x + y).$$

So, $T((1,\, 1)) = (4 - 2,\, 2 + 1)$

$= (2,\, 3)$

$= 3(1,\, 1) + 1(-1,\, 0)$

$\Rightarrow T(1,\, 1) \;= 3(1,\, 1) + 1(-1,\, 0)$

Similarly $T(-1,\, 0) \;= (-4,\, -2)$

$= -2(1,\, 1) + 2(-1,\, 0)$ *Therefore the matrix corresponding to the operator* T *is given by* $m(T) = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$.

(ii) Let T be the operator corresponding to the matrix $\begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$ *and* $(x,\, y) \in \mathbb{R}^2$*. Then* $(x,\, y) = \alpha_1(1,\, 1) + \alpha_2(-1,\, 0)$

$= (\alpha_1 - \alpha_2,\, \alpha_1)$

$\Rightarrow \alpha_1 = y,\, \alpha_2 = y - x$

$\Rightarrow (x,\, y) = y(1,\, 1) + (y - x)(-1,\, 0)$

Now

$$T(x,\, y) = T\left(\sum_{i=1}^{2} \alpha_i e_i\right) = \sum_{i=1}^{2} \alpha_i T(e_i)$$

$$T(x,\, y) = \sum_{i=1}^{2} \alpha_i \sum_{j=1}^{2} a_{ji} e_j = \sum_{j=1}^{2} (\sum_{i=1}^{2} a_{ji}\alpha_i) e_j$$

$$T\left(\sum_{i=1}^{2} \alpha_i e_i\right) = \sum_{j=1}^{2}\left(\sum_{i=1}^{2} a_{ji}\alpha_i\right) e_j \quad \text{here } \alpha_1 = y,\, \alpha_2 = y - x$$

So,

$$\begin{aligned}
T(x,\, y) &= \sum_{j=1}^{2} (a_{j1}\alpha_1 + a_{j2}\alpha_2) e_j \\
&= (a_{11}\alpha_1 + a_{12}\alpha_2)e_1 + (a_{21}\alpha_1 + a_{22}\alpha_2)e_2 \\
&= (3y - 2(y - x))(1,\, 1) + (y + 2(y - x))(-1,\, 0) \\
&= (y + 2x)(1,\, 1) + (3y - 2x)(-1,\, 0) \\
&= (y + 2x - 3y + 2x,\, y + 2x) \\
&= (4x - 2y,\, y + 2x) \\
\Rightarrow T(x,\, y) &= (4x - 2y,\, y + 2x)
\end{aligned}$$

Hence verified.

2. *Find the matrix representation of* $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ *defined as* $T(x,\, y) =$

$(3x-4y,\, x+5y)$ *with respect to the basis* $B=\{(1,\,0),\,(0,\,1)\}$.

Solution *We have a linear operator* T *on* V *given by* $T(x,\,y)=(3x-4y,\,x+5y)$. *So,* $T(1,\,0)=(3,\,1)$

$$=\alpha(1,\,0)+\beta(0,\,1)$$

$$=(\alpha,\,\beta)$$

$\Rightarrow(3,\,1)=(\alpha,\,\beta)$

$\Rightarrow\alpha=3,\,\beta=1$

$T(1,\,0)=3(1,\,0)+1(0,\,1)$.

Similarly $T(0,\,1)=(-4,\,5)$

$=-4(1,\,0)+5(0,\,1)$.

Therefore the matrix corresponding to the linear operator T *is given by* $m(T)=\begin{bmatrix}3 & -4\\ 1 & 5\end{bmatrix}$.

3. *Let* $V=W=F_n[x]$ *be the vector space of all polynomials of degree* $\leq n$. *Define a linear transformation* $T:V\to W$ *by* $T(f)=f'$. *Choose the basis* $\{x_1,\,x_2,\,x_3,\,\ldots\}$ *is a basis of* V *(and* W*). Then* $T(1)=0$, $T(x)=1$, $T(x^2)=2x,\,\ldots,\,T(x^n)=nx^{n-1}$.

$T(1)=0=01+0x+\ldots+0x^n$

$T(x)=1=1+0x+0x^2+\ldots+0x^n$

$T(x^2)=2x=01+2x+0x^2+\ldots+0x^n$

$\vdots$

$T(x^n)=nx^{n-1}=01+0x+\ldots+nx^{n-1}+0x^n$.

Therefore, the matrix corresponding to T *is given by*

$$m(T)=\begin{bmatrix}0 & 1 & 0 & \ldots & 0 & 0\\ 0 & 0 & 2 & \ldots & 0 & 0\\ \vdots & & & & & \\ 0 & 0 & 0 & \ldots & n & 0\end{bmatrix}.$$

4. *Let* $V = \mathbb{R}^3$ *and let* $T : V \to V$ *be the linear transformation defined by* $T(x,\, y,\, z) = (2x,\, 4y,\, 5z)$*. Find the matrix of* T *with respect to the basis* $\{(\frac{2}{3},\, 0,\, 0),\, (0,\, \frac{1}{2},\, 0),\, (0,\, 0,\, \frac{1}{4})\}$ *of* V*.*

Solution *We have the given linear transformation* $T(x,\, y,\, z) = (2x,\, 4y,\, 5z)$*. So,*

$$\begin{aligned}
T\left(\frac{2}{3},\, 0,\, 0\right) &= \left(\frac{4}{3},\, 0,\, 0\right) = 2\left(\frac{2}{3},\, 0,\, 0\right) + 0\left(0,\, \frac{1}{2},\, 0\right) + 0\left(0,\, 0,\, \frac{1}{4}\right)\\
T\left(0,\, \frac{1}{2},\, 0\right) &= (0,\, 2,\, 0) = 0\left(\frac{2}{3},\, 0,\, 0\right) + 4\left(0,\, \frac{1}{2},\, 0\right) + 0\left(0,\, 0,\, \frac{1}{4}\right)\\
T\left(0,\, 0,\, \frac{1}{4}\right) &= \left(0,\, 0,\, \frac{5}{4}\right) = 0\left(\frac{2}{3},\, 0,\, 0\right) + 0\left(0,\, \frac{1}{2},\, 0\right) + 5\left(0,\, 0,\, \frac{1}{4}\right).
\end{aligned}$$

Therefore the matrix corresponding to T *is given by*

$$m(T) = \begin{bmatrix} 2 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 5 \end{bmatrix}.$$

5. *Let* $V = \mathbb{R}^3$ *and let* $A = \begin{bmatrix} 1 & 2 & 3\\ 3 & 1 & -5\\ 0 & 0 & 1 \end{bmatrix}$ *be the matrix of* $T \in L(V,\, V)$ *with respect to the basis* $\{(1,\, 0,\, 0),\, (0,\, 1,\, 0),\, (0,\, 0,\, 1)\}$*. Find the matrix of* T *with respect to the basis* $\{(1,\, 1,\, 0),\, (0,\, 1,\, 0),\, (0,\, 1,\, 1)\}$*.*

Solution *Let* $(x,\, y,\, z) \in V$ *be any element and* $T \in L(V,\, V)$*. Then* $(x,\, y,\, z) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$*, where* $\alpha_1 = x,\, \alpha_2 = y,\, \alpha_3 = z$*,*
$e_1 = (1,\, 0,\, 0),\, e_2 = (0,\, 1,\, 0,\, e_3 = (0,\, 0,\, 1)$*.*

Now the linear transformation corresponding to A *w.r.t. the basis* $\{e_1,\, e_2,\, e_3\}$ *is given by* $T(x,\, y,\, z) = \begin{bmatrix} 1 & 2 & 3\\ 3 & 1 & -5\\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} x + 2y + 3z\\ 3x + y - 5z\\ z \end{bmatrix}$

$\Rightarrow T(x,\, y,\, z) = (x + 2y + 3z,\, 3x + y - 5z,\, z)$.

To find the matrix of T *w.r.t. the basis* $\{(1,\, 1,\, 0),\, (0,\, 1,\, 0),\, (0,\, 1,\, 1)\}$, *we have*

$T(1,\, 1,\, 0) = (3,\, 4,\, 0) = \alpha(1,\, 1,\, 0) + \beta(0,\, 1,\, 0) + \gamma(0,\, 1,\, 1)$

$$(\alpha,\, \alpha + \beta + \gamma,\, \gamma) = (3,\, 4,\, 0)$$

$$\Rightarrow \alpha = 3, \gamma = 0, \beta = 1$$

$$\Rightarrow T(1,\, 1,\, 0) = 3(1,\, 1,\, 0) + 1(0,\, 1,\, 0) + 0(0,\, 1,\, 1).......(i)$$

$$T(0,\, 1,\, 0) = \alpha(1,\, 1,\, 0) + \beta(0,\, 1,\, 0) + \gamma(0,\, 1,\, 1)$$

$$(2,\, 1,\, 0) = (\alpha,\, \alpha + \beta + \gamma,\, \gamma)$$

$$\alpha = 2,\, \gamma = 0,\, \beta = -1$$

$$T(0,\, 1,\, 0) = 2(1,\, 1,\, 0) - 1(0,\, 1,\, 0) + 0(0,\, 1,\, 1).........(ii)$$

$$\textit{Also } T(0,\, 1,\, 1) = \alpha(1,\, 1,\, 0) + \beta(0,\, 1,\, 0) + \gamma(0,\, 1,\, 1)$$

$$(5,\, -4,\, 1) = (\alpha,\, \alpha + \beta + \gamma,\, \gamma)$$

$$\alpha = 5,\, \gamma = 1,\, \beta = -5$$

$$\Rightarrow T(0,\, 1,\, 1) = 5(1,\, 1,\, 0) - 5(0,\, 1,\, 0) + 1(0,\, 1,\, 1).......(iii)$$

From equations (i), (ii), *and* (iii) *we get the matrix corresponding to* T *as*

$$m(T) = \begin{bmatrix} 3 & 2 & 5 \\ 1 & -1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

(10.5) Let Us Sum Up: *Matrix is a vector and linear transformation is a mapping. In this lesson we got the result there is one to one correspondence between set of linear transformations on finite dimensional vector spaces and the set of matrices. One can easily understand this correspondence through various examples done in this lesson and operations on matrices with the help* *of operations on linear transformation.*

(10.6) Lesson End Exercise

1. *Let* T *be a linear operator on* $\mathbb{R}^3$ *defined by* $T(x,\, y,\, z) = (2y+z,\, x-4y,\, 3x)$. *Find the matrix of* T *with respect to basis* $\{(1,\, 1,\, 1),\, (1,\, 1,\, 0),\, (1,\, 0,\, 0)\}$ *and verify it with the linear transformation.*

2. *Find the matrix representation of* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *defined as* $T(x,\, y) = (3x - 4y,\, x + 5y)$ *with respect to the basis* $B = \{(1,\, 0),\, (0,\, 1)\}$.

3. *Given the matrix* $\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{2}{3} & 4 \end{bmatrix}$. *Find the linear operator* T *on* $\mathbb{R}^2$ *with respect to basis* $\{(1,\, 0),\, (1,\, 1)\}$ *corresponding to the given matrix.*

4. *Let* T *be a linear operator on* $\mathbb{R}^3$ *defined by* $T(x,\, y,\, z) = (2x - 3y + 4z,\, 5x - y + 2z,\, 4x + 7y)$. *Find the matrix of* T *with respect to basis* $\{(1,\, 0,\, 0),\, (0,\, 1,\, 0),\, (0,\, 0,\, 1)\}$.

Answers $(i) T(x,\, y,\, z) = (y + z,\, x - y,\, -x - y)$ (ii) $\begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & -4 \\ 1 & 5 \end{bmatrix}$ (iv) $T(x,\, y) = \left(\frac{7x+23y}{6},\, \frac{2x+10y}{3}\right)$ (v) $\begin{bmatrix} 2 & -3 & 4 \\ 5 & -1 & 2 \\ 4 & 7 & 0 \end{bmatrix}$.

(10.7) Model University Questions

1. *If matrix of linear operator* T *on* $\mathbb{R}^3$ *with respect to basis* $\{(1,\, 0,\, 0),\, (0,\, 1,\, 0),\, (0,\, 0,\, 1)\}$ *is* $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$. *Then what is the matrix*

of T with respect to basis $\{(0, 1, -1), (1, -1, 1), (-1, 1, 0)\}$.

2. *If the matrix of linear operator T on* $\mathbb{R}^3$ *with respect to the standrad basis is* $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$. *Find the matrix corresponding to the basis* $\{(1, 2, 2), (1, 1, 2), (1, 2, 1)\}$.

(10.8) Suggested Readings:*(i) N.S. Gopalakrishnan, University Algebra, New Age International (P) Limited, Publishers.*

(ii) Kenneth Hoffman, Ray Kunze, Linear Algebra,Prentice Hall India.

(iii) Singh, S. and Zameerudin, Q.,Modern Algebra, Vikas Publishing House Pvt.Ltd

Lesson-XI Kernel and Range of Linear Transformation

(11.1) Introduction: *If V and W are vector spaces over the same field F and $T : V \to W$ is a linear transformation. Then we look into the subspaces in V and V' and they turn out be in the form of kernel and image of T which we explain in detail in this lesson. These two concepts are analogus to the kernel and image of group homomorphism or ring homomorphism*

(11.2) Objective *: The properties of linear transformation become easy to understand through kernel and image of linear transformation.*

(11.3) Kernel and Range of linear transformation

(11.3.1) Definition *: Let V, W be vector spaces over a field F and $T : V \to W$ be a linear transformation. Then the subset $\{x \in V | T(x) = 0'\}$ of V is said to be a* **kernel of** *T. It is denoted by $Ker(T)$ and $Ker(T) = \{x \in V | T(x) =$*

$0'\}$. *It is also called as* **Null space** *of* T.

(11.3.2) Definition:(Range of linear transformation): *Let* V, W *be vector spaces over a field* F *and* $T : V \to W$ *be a linear transformation. Then the image of set* V *under* T *is called* **Range** *of* T *i.e.*

$$Range(T) = \{w \in W | w = T(v) \text{ for some } v \in V\}.$$

(11.3.3) Theorem : *Let* V, W *be vector spaces over a field* F *and* $T : V \to W$ *be a linear transformation.Then* $Ker(T)$ *and* $Range(T)$ *are subspaces of* V *and* W *respectively.*

Proof *Since* $T(0) = 0'$. *So* $0 \in Ker(T) \Rightarrow Ker(T) \neq \phi$.

Now, let $x, y \in Ker(T)$ *and* $\alpha, \beta \in F$.

Then $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) = \alpha 0' + \beta 0' = 0'$

$\Rightarrow \alpha x + \beta y \in Ker(T)$.

Therefore, $Ker(T)$ *is a subspace of* V.

Similarly $Range(T) \neq \phi$ *because* $(T(0) = 0')$.

Let $x', y' \in W$ *and* $\alpha, \beta \in F$. *Then there exists* $x, y \in V$ *such that* $x' = T(x)$ *and* $y' = T(y)$.

Now $\alpha x' + \beta y' = \alpha T(x) + \beta T(y) = T(\alpha x + \beta y)$

$\Rightarrow \alpha x' + \beta y' \in Range(T)$. *Hence* $Range(T)$ *is a subspace of* W.

(11.4) Rank and Nullity of linear transformation

(11.4.1) Definition: *Let* V, W *be vector spaces over a field* F *and* $T : V \to W$ *be a linear transformation. Then the dimension of the* $Range(T)$ *is called the* **rank of** T *and the dimension of* $Ker(T)$ *is called the* **Nullity** *of* T.

(11.4.2) Theorem (Rank-Nullity Theorem): *Let* V, W *be vector spaces over a field* F *and* $T : V \to W$ *be a linear transformation. Suppose the dimension of* V *is* n, *then*

$$\dim V = Rank(T) + Nullity(T).$$

Proof *Let $Nullity(T) = m$. Then $m \leq n$ because $Ker(T)$ is a subspace of V. Now, suppuse that $B = \{x_1, x_2, \ldots, x_m\}$ is a basis of $Ker(T)$. Then B is a linearly independent subset of V. Therefore, by Basis-Extension theorem B can be extended to a basis of V. Let $B_1 = \{x_1, x_2, \ldots, x_m, x_{m+1}, x_n\}$ be a basis of V. Cosider the set $B_2 = \{T(x_{m+1}), T(x_{m+2}), \ldots, T(x_n)\}$.*

Claim that B_2 is a basis of $Im(T)$.

For this, Consider $a_{m+1}T(x_{m+1}) + a_{m+2}T(x_{m+2}) + \cdots + a_nT(x_n) = 0'$

$T(a_{m+1}x_{m+1} + a_{m+2}x_{m+2} + \ldots + a_nx_n) = 0'$

$a_{m+1}x_{m+1} + a_{m+2}x_{m+2} + \ldots + a_nx_n \in Ker(T)$

$\Rightarrow a_{m+1}x_{m+1} + a_{m+2}x_{m+2} + \ldots + a_nx_n = a_1x_1 + a_2x_2 + \ldots + a_mx_m$

$\Rightarrow a_1x_1 + a_2x_2 + \ldots + a_mx_m - a_{m+1}x_{m+1} - a_{m+2}x_{m+2} - \ldots + a_nx_n = 0$

$\Rightarrow a_1 = a_2 = \ldots = a_m = a_{m+1} = \ldots = a_n = 0$

$\Rightarrow a_{m+1} = \ldots = a_n = 0.$

Hence B_2 is linearly independent.

Now, let $y \in Im(T)$. Then there exists $x \in V$ such that $T(x) = y$

$\Rightarrow T(a_1x_1 + a_2x_2 + \ldots + a_mx_m + \ldots + a_nx_n) = y$

$\Rightarrow a_1T(x_1) + a_2T(x_2) + \ldots + a_mT(x_m) + a_{m+1}T(x_{m+1}) + \ldots a_nT(x_n) = y$

$\Rightarrow 0' + a_{m+1}T(x_{m+1}) + \ldots a_nT(x_n) = y$

$\Rightarrow y = a_{m+1}T(x_{m+1}) + \ldots a_nT(x_n).$

Therefore $y \in L(B_2) \Rightarrow V = L(B_2)$. Hence B_2 is a basis of $Im(T)$. This shows that $Rank(T) = \dim(Im(T)) = n - m = \dim V - \dim(Ker(T))$

$\Rightarrow \dim V = Nullity(T) + Rank(T).$

(11.4.3) Theorem *: Let V, W be vector spaces over a field F and $T : V \to W$ be a linear transformation. Then T is one-one if and only if $Ker(T) = 0$.*

Proof *First, we suppose that T is one-one mapping.*

Now, let $x \in Ker(T)$. Then $T(x) = 0'$

$\Rightarrow T(x) = T(0)$

$\Rightarrow x = 0$. *Hence* $Ker(T) = \{0\}$.

Conversely, Suppose that $Ker(T) = \{0\}$. *To show that* T *is one-one, cosider* $T(x_1) = T(x_2)$

$\Rightarrow T(x_1) - T(x_2) = 0'$

$\Rightarrow T(x_1 - x_2) = 0'$

$\Rightarrow x_1 - x_2 \in Ker(T)$. *But* $Ker(T) = \{0\}$

$\Rightarrow x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$. *Hence* T *is one-one.*

(11.4.4) Theorem (First fundamental theorem of isomorphism): *Let* V, W *be vector spaces over a field* F *and* $T : V \to W$ *be an onto linear transformation. Then* $V/Ker(T) \cong W$.

Proof *Define a rule* $\overline{T} : V/K \to W$ *by* $\overline{\scriptstyle T(x+K)=T(x), \forall x \in V}$, *where* $K = Ker(T)$.

I $\overline{T}$ is well defined mapping: *Let* $x_1 + K = x_2 + K$

$\Rightarrow x_1 - x_2 \in K$

$\Rightarrow T(x_1 - x_2) = 0'$

$\Rightarrow T(x_1) = T(x_2)$

$\Rightarrow \overline{T}(x_1 + K) = \overline{T}(x_2 + K)$

$\Rightarrow \overline{T}$ *is a well-defined map.*

II $\overline{T}$ is one-one:

$Ker(\overline{T}) = \{x + K | \overline{T}(x + k) = 0'\}$

$= \{x + K | T(x) = 0'\}$

$= \{x + K | x \in K\}$

$= \{K\}$

$Ker(\overline{T}) = \{K\} \Rightarrow \overline{T}$ *is one-one.*

III $\overline{T}$ is linear transformation: *Let* $x + K$, $y + K \in V/W$ *and* α, $\beta \in F$.

Then $\overline{T}(\alpha(x+K)+\beta(y+K))=\overline{T}((\alpha x+\beta y)+K))$

$=T(\alpha x+\beta y)$

$=\alpha T(x)+\beta T(y)$

$=\alpha\overline{T}(x+K)+\beta\overline{T}(y+K)$

$\Rightarrow \overline{T}(\alpha(x+K)+\beta(y+K))=\alpha\overline{T}(x+K)+\beta\overline{T}(y+K)$.

IV $\overline{T}$ is Onto *Since T is onto, so $\overline{T}$ is clearly onto.Hence $V/K\cong W$.*

(11.5) Examples

1. *Show that $T:\mathbb{R}^3\to\mathbb{R}^3$ defined by $T(x,\,y,\,z)=(x-y,\,x+2y,\,y)$ is a linear transformation. Determine (i)$T(e_1)$, $T(e_2)$ and (ii)$Ker(T)$.*

Solution *Here* $T(0,\,0,\,0)=(0-0,\,0+2.0,\,0)=(0,\,0,\,0)$. *Now, let* $(x_1,\,y_1,\,z_1),\,(x_2,\,y_2,\,z_2)\in\mathbb{R}^3$ *and* $\alpha,\,\beta\in F$. *Then*

$T(\alpha(x_1,\,y_1,\,z_1)+\beta(x_2,\,y_2,\,z_2))=T(\alpha x_1+\beta x_2,\,\alpha y_1+\beta y_2,\,\alpha z_1+\beta z_2)$

$=(\alpha x_1+\beta x_2-\alpha y_1-\beta y_2,\,\alpha x_1+\beta x_2+2(\alpha y_1+\beta y_2),\,\alpha y_1+\beta y_2)$

$=(\alpha(x_1-y_1)+\beta(x_2-y_2),\,\alpha(x_1+2y_1)+\beta(x_2+2y_2),\,\alpha y_1+\beta y_2)$

$=(\alpha(x_1-y_1),\,\alpha(x_1+2y_1),\,\alpha y_1)+(\beta(x_2-y_2),\,\beta(x_2+2y_2),\,\beta y_2)$

$=\alpha\,((x_1-y_1),\,(x_1+2y_1),\,y_1)+\beta\,((x_2-y_2),\,(x_2+2y_2),\,y_2)$

$=\alpha T(x_1,\,y_1,\,z_1)+\beta T(x_2,\,y_2,\,z_2)$

$\Rightarrow T$ *is a linear transformation.*

Now, $T(e_1)=T(1,\,0,\,0)=(1,\,1,\,0)$ *and* $T(0,\,1,\,0)=(-1,\,2,\,1)$.

Kernel of T is given by

$Ker(T)=\{(x,\,y,\,z)|T(x,\,y,\,z)=(0,\,0,\,0)\}$

$=\{(x,\,y,\,z)|(x-y,\,x+2y,\,y)=(0,\,0,\,0)\}$

$=\{(x,\,y,\,z)|x-y=0,\,x+2y=0,\,y=0\}$

$=\{(x,\,y,\,z)|x=y=0\}$

$=\{(0,\,0,\,z)|z\in F\}$

$\Rightarrow Ker(T)=\{(0,\,0,\,z)|z\in F\}$.

2. *Let V be a finite dimensional vector space over a field F and T be linear operator on V. Then T is one-one if and only if T is onto.*

Solution *Suppose that T is one-one.*

Then $Ker(T) = \{0\} \Rightarrow Nullity(T) = 0$. Therefore, by Rank-Nullity theorem, we have $\dim V = Nullity(T) + Rank(T) = 0 + Rank(T)$

$\Rightarrow V \cong Range(T)$

$\Rightarrow V = Range(T) \Rightarrow T$ *is onto.*

Conversely, suppose that T is onto. Then $V \cong Range(T) \Rightarrow$ $\dim V = \dim Range(T) = Rank(T)$. *Now by Rank-Nullity theorem, we have* $\dim V = Nullity(T) + Rank(T)$

$\Rightarrow \dim V = Nullity(T) + \dim V$

$\Rightarrow Nullity(T) = 0$

$\Rightarrow Ker(T) = \{0\} \Rightarrow T$ *is one-one.*

3. *For each of the following transformations $T : V \to W$. Find a basis and dimension of its (i)Range space (ii) Null space. Also verify the Rank-Nullity Theorem*

(a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by* $T(x,\, y,\, z) = (x + 2y - z,\, y + z,\, x + y - 2z)$

$(b)T : \mathbb{R}^2 \to \mathbb{R}^3$ *be defined by* $T(x,\, y) = (x - y,\, y - x,\, x)$

$(c)T : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by* $T(x,\, y) = (x + y,\, x - y)$

Solution *(a) Since* $\{(1,\, 0,\, 0),\, (0,\, 1,\, 0),\, (0,\, 0,\, 1)\}$ *is a basis of* $\mathbb{R}^3$ *so*

$$\{T(1,\, 0,\, 0),\, T(0,\, 1,\, 0),\, T(0,\, 0,\, 1)\} = \{(1,\, 0,\, 1),\, (2,\, 1,\, 1),\, (-1,\, 1,\, -2)\}$$

generates Range(T). Consider $\alpha(1,\, 0,\, 1) + \beta(2,\, 1,\, 1) + \gamma(-1,\, 1,\, -2) = (0,\, 0,\, 0)$

$(\alpha + 2\beta - \gamma,\, \beta + \gamma,\, \alpha + \beta - 2\gamma) = (0,\, 0,\, 0)$

$\Rightarrow \alpha + 2\beta - \gamma = 0$

$\beta + \gamma = 0$

$\alpha + \beta - 2\gamma = 0$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Now \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 0$$

$\Rightarrow \{(1,\, 0,\, 1),\, (2,\, 1,\, 1),\, (-1,\, 1,\, -2)\}$ *is L.D.*

$$Let\ A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \quad operate\ R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \quad operate\ \ R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This shows that $\{(1,\, 2,\, -1),\, (0,\, 1,\, 1)\}$ *is a basis of* $Range(T)$

$\Rightarrow Rank(T) = 2.$

Now, let $(x,\, y,\, z) \in Ker(T) \Rightarrow T(x,\, y,\, z) = (0,\, 0,\, 0)$

$\Rightarrow (x + 2y - z,\, y + z,\, x + y - 2z) = (0,\, 0,\, 0)$

$\Rightarrow x + 2y - z = 0$

$y + z = 0$

$x + y - 2z = 0$

$\Rightarrow x = 3,\, y = -1,\, z = 1$

Therefore $Ker(T)$ *is generated by* $\{(3,\, -1,\, 1)\} \Rightarrow \{(3,\, -1,\, 1)\}$ *is a basis of* $Ker(T)$. *Hence* $Nullity(T) = 1$ *and* $\dim(\mathbb{R}^3) = 1 + 2 = Nullity(T) + Rank(T)$

which shows that Rank-Nullity theorem is verified.

(b) *(Do yourself)*

(c) *(Do yourself)*

(11.6) Let Us Sum Up *:This lesson deals with the most important theorem of Sylvester* **Rank-Nullity theorem**. *We have defined the kernel and range of linear transformation, then illustrated these concepts with examples. Some important properties of linear transformation have also been observed through kernel and range of linear transformation.*

(11.7)Lesson End Exercise

1. *Show that mapping* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by*
$T(x,\, y,\, z) = (y + z,\, x + y - 2z,\, x + 2y - 2z)$ *is a linear transformation. Find* $Range(T),\, Ker(T),\, Rank(T),\, Nullity(T)$.

Ans $Range(T) = L((1,\, 1,\, 2),\, (0,\, 1,\, 1)),\, Ker(T) = L((3,\, -1,\, 1))$

2. *Find a linear transformation* $T : \mathbb{R}^2 \to \mathbb{R}^3$
such that $T(1,\, 2) = (3,\, -1,\, 5)$ *and* $T(0,\, 1) = (2,\, 1,\, -1)$. *Also find* $Range(T),\, Ker(T),\, Rank(T),\, Nullity$.

Ans $Range(T) = L((2,\, 1,\, -1),\, (-1,\, -3,\, 7)),\, Ker(T) = \{(0,\, 0,\, 0)\}$

3. *Find a linear transformation* $T : \mathbb{R}^3 \to \mathbb{R}^2$ *whose image is generated by* $\{(1,\, 2,\, 3),\, (4,\, 5,\, 6)\}$.

Hint *Since* $B = \{(1,\, 0,\, 0),\, (0,\, 1,\, 1),\, (0,\, 0,\, 1)\}$ *is a usual basis of* $\mathbb{R}^3$. *So, the range of* T *is generated by* $T(e_1),\, T(e_2),\, T(e_3)$. *Put* $T(e_1) = (1,\, 2,\, 3),\, T(e_2) = (4,\, 5,\, 6)$ *and* $T(e_3) = (0,\, 0,\, 0)$.

Now let $(x,\, y,\, z) \in \mathbb{R}^3$. *Then* $(x,\, y,\, z) = xe_1 + ye_2 + ze_3$

$\Rightarrow T(x,\, y,\, z) = xT(e_1) + yT(e_2) + zT(e_3)$

$= x(1,\, 2,\, 3) + y(4,\, 5,\, 6)$

$= (x + 4y,\, 2x + 5y,\, 3x + 6y)$.

(11.8) University Model Questions

1. *Let V and W be two vector spaces over the same field F and $T : V \to W$ be a linear transformation with kernel K. Prove that*
$(i)K$ is a subspace of V.
$(ii)T(V)$ is a subspace of W.
2. *Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose image is generated by* $(1, 0, -1), (1, 2, 2)$.
3. *Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a mapping defined as*
$T(x, y, z) = (x, y), \forall (x, y, z) \in \mathbb{R}^3$. *Show that T is a linear transformation and find $Ker(T)$.*

(11.9) Suggested Readings *:(i) N.S. Gopalakrishnan, University Algebra, New Age International (P) Limited, Publishers.*
(ii) Kenneth Hoffman, Ray Kunze, Linear Algebra,Prentice Hall India.
(iii) Singh, S. and Zameerudin, Q., Modern Algebra, Vikas Publishing House Pvt. ltd.

Lesson-XII Inverse of Linear Transformation

12.0 **Structure**

(12.1) Introduction *: In this lesson we assume linear transformations on finite dimensional vector spaces. Analogous to the inverse of functions, we can find inverse of bijective linear transformations. Moreover the inverse of a linear transformation also turns out to be a bijective linear transformation.*

(12.2) Objective *: The students will understand the techniques of explicit computation of the inverse of a bijective linear transformation.*

(12.3) Bijective linear transformation

(12.3.1) Definition (One-One Transformation)*: Let $T : V \to W$ be a linear transformation. Then T is said to be one-one if $T(x) = T(y) \Rightarrow x = y, \forall\ x, y \in V$.*

(12.3.2) Definition (Onto Transformation)*: Let $T : V \to W$ be a linear* *transformation. Then T is said to be onto if $W = Range(T)$.*

(12.3.3) Definition (Bijective Transformation)*: Let $T : V \to W$ be a linear transformation. Then T is bijective if it is both one-one and onto.*

(12.3.4) Definition (Non Singular Transformation)*: A linear transformation $T : V(F) \to W(F)$ is said to be non- singular if $Ker(T) = \{0\}$.*

(12.3.5) Definition (Singular Transformation)*: A linear transformation $T : V(F) \to W(F)$ is said to be singular if $Ker(T) \neq \{0\}$.*

(12.3.6) Theorem*: A linear transformation $T : V \to W$ is non singular if and only if the images of a linearly independent set is linearly independent.*

Proof *Suppose that $T : V \to W$ is non singular. Let $\{x_1, x_2, \ldots, x_n\}$ be a linearly independent subset of V. We have to show that $\{T(x_1), T(x_2), \ldots, T(x_n)\}$ is linearly independent.*

For this, consider $\alpha_1 T(x_1) + \alpha_2 T(x_2) + \ldots + \alpha_n T(x_n) = 0$

$\Rightarrow T(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n) = 0$

$\Rightarrow \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0.$

Therefore $\{T(x_1), T(x_2), \ldots, T(x_n)\}$ is linearly independent.

Conversely, suppose that T takes linearly independent subset to a linearly independent. Let $x \in Ker(T)$. Then $T(x) = 0$. To prove that T is non-singular we have to show that $x = 0$. For this, suppose that $x \neq 0$. Then $\{x\}$ is linearly independent which implies that $\{T(x)\}$ is linearly independent. Therefore $T(x) \neq 0$ which is a contradiction. $\Rightarrow x = 0$.

Hence T is non singular.

(12.4) Invertible Operator

(12.4.1) Definition *: A linear operator $T : V \to V$ is said to be invertible operator if there exists an operator $S : V \to V$ such that $ST = TS = I$, where*

I is an identity operator. Here S is called the inverse of T and is denoted by T^{-1}.

(12.4.2) Theorem: *The inverse of linear operator is unique.*

Proof *Let $T : V \to V$ be an invertible operator. If possible, suppose that there exist two inverse of T say S_1, S_2. Then $S_1T = I = TS_1$......(1)*

$S_2T = I = TS_2$.......(2).

Now $S_1 = S_1I$

$$= S_1(TS_2)$$

$$= (S_1T)S_2$$

$$= IS_2$$

$$= S_2.$$

Therefore, the inverse of an invertible operator must be unique.

(12.4.3) Theorem: *Let V be a vector space over a field F and $T : V \to V$ be a linear operator. Then T is invertible if and only if T is bijective.*

Proof *First, we suppose that T is bijective. To prove that T is invertible, we define $S : V \to V$ by $S(y) = x$ if $y = T(x)$.*

I S is well defined function: *Since T is one-one, onto. So for each $y \in V$ there exixts a unique $x \in V$ such that $y = T(x) \Rightarrow$ there exists a unique $x \in V$ such that $S(y) = x$.*

Therefore S is a well defined map.

II S is linear operator: *Let y_1, $y_2 \in V$ and $S(y_1) = x_1$ so that $y_1 = T(x_1)$*

$S(y_2) = x_2$ so that $y_2 = T(x_2)$.

Let α, $\beta \in F$. Then $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2) = \alpha y_1 + \beta y_2$

$\Rightarrow \alpha x_1 + \beta x_2 = S(\alpha y_1 + \beta y_2)$

$\Rightarrow \alpha S(y_1) + \beta S(y_2) = S(\alpha y_1 + \beta y_2)$.

Therefore, S is a linear operator.

Now, let $x \in V$ *and* $y = T(x)$. *Then* $S(y) = x$.

$(ST)(x) = S(T(x)) = S(y) = x$

$\Rightarrow ST = I$.

Similarly, $TS(y) = T(S(y)) = T(x) = y$

$\Rightarrow TS = I$.

Hence T *is invertible and* $T^{-1} = S$.

Conversly, suppose that T *is invertible. Then there exists a linear operator* $S : V \to V$ *such that* $ST = TS = I$. *Now, let* x, $y \in V$ *such that* $T(x) = T(y)$

$\Rightarrow S(T(x)) = S(T(y))$

$\Rightarrow (SoT)(x) = (SoT)(y)$

$\Rightarrow I(x) = I(y)$

$\Rightarrow x = y$. *Therefore,* T *is one-one.*

Similarily, let $y \in V$ *Then* $S(y) = x \Rightarrow T(x) = y$. *So there exists* $x \in V$ *such that* $T(x) = y \Rightarrow T$ *is onto. Hence* T *is one-one and onto.*

(12.4.4) Theorem: *If* T, S, U *be linear operators on* V *such that* $ST = TU = I$. *Then* T *is invertible and* $S = U = T^{-1}$.

Proof *Given that* T, S, U *are linear operators on* V *such that*

$ST = TU = I$.

To show that T *is invertible, it is enough to show that* T *is one-one and onto.*

For this, let x, $y \in V$ *such that* $T(x) = T(y)$

$\Rightarrow S(T(x)) = S(T(y))$

$\Rightarrow ST(x) = ST(y)$

$\Rightarrow I(x) = I(y)$

$\Rightarrow x = y$. *Thus* T *is one-one.*

Now, let $y \in V$ *be any element, then there exists* $x \in V$ *such that* $U(y) = x$ *because* S *is a mapping.*

$\Rightarrow T(x) = T(U(y)) = TU(y) = I(y) = y$

$\Rightarrow T$ *is onto. Hence* T *is invertible.*

Now we show that $S = U = T^{-1}$, *for that we have* $ST = I$

$\Rightarrow (ST)T^{-1} = IT^{-1}$

$\Rightarrow S(TT^{-1}) = T^{-1}$

$\Rightarrow SI = T^{-1}$

$\Rightarrow S = T^{-1}$ *Also we have* $TU = I$

$\Rightarrow T^{-1}(TU) = T^{-1}I$

$\Rightarrow (T^{-1}T)U = T^{-1}$

$\Rightarrow IU = T^{-1}$

$\Rightarrow U = T^{-1}$. *Hence the result.*

(12.4.5) Theorem*: Let* V *be a vector space over a field* F *and* T, S *be linear operators on* V. *Then*

(i) *if* S *and* T *are invertible, then* TS *is also invertible and* $(TS)^{-1} = S^{-1}T^{-1}$.

(ii) *if* T *is invertible and* $0 \neq \alpha \in F$, *then* αT *is invertible and* $(\alpha T)^{-1} = \frac{1}{\alpha}T^{-1}$

(iii) *if* T *is invertible, then* T^{-1} *is also invertible and* $(T^{-1})^{-1} = T$.

Proof (i) *Given that* S, T *are invertible*$\Rightarrow$ *There exists* S^{-1}, T^{-1} *such that* $SS^{-1} = S^{-1}S = I$ *and* $TT^{-1} = T^{-1}T = I$.

To show that ST *is invertible, first we show that* ST *is one-one: Cosider* $ST(x) = ST(y)$

$\Rightarrow S(T(x)) = S(T(y))$

$\Rightarrow T(x) = T(y)$ *because* S *is one-one*

$\Rightarrow x = y$ *because* T *is one-one* .

Therefore ST *is one-one.*

Now, let $y \in V$ *be any element. Since* S *is onto, so there extists* $x \in V$ *such that* $S(x) = y$. *Similarly* T *is onto so for each* $x \in V$ *there exists* $z \in V$

such that $x = T(z)$. *Therefore for each* $y \in V$ *there exists* $z \in V$ *such that* $ST(z) = S(T(z)) = S(x) = y$.

Hence ST *is invertible. Now,* $(ST)(T^{-1}S^{-1}) = S(TT^{-1})S^{-1} = SIS^{-1} = SS^{-1} = I$

Similarly $(T^{-1}S^{-1})(ST) = T^{-1}(S^{-1}S)T = T^{-1}IT = T^{-1}T = I$

$\Rightarrow (ST)^{-1} = T^{-1}S^{-1}$.

(ii) To show that αT *is invertible: For let* $(\alpha T)(x) = (\alpha T)(y)$

$\Rightarrow \alpha T(x) = \alpha T(y)$

$\Rightarrow T(x) = T(y)$ *because* $\alpha \neq 0$ *in* F

$\Rightarrow x = y$. *Therefore* αT *is one-one.*

Let $y \in V$. *Then there exists* $x \in V$ *such that* $T(x) = y$ *as* T *is onto. This implies that for each* $y \in V$ *there exists* $\frac{1}{\alpha}x$ *such that* $(\alpha T)(\frac{1}{\alpha}x) = \alpha\frac{1}{\alpha}T(x) = T(x) = y \Rightarrow \alpha T$ *is onto. Hence* αT *is invertible.*

Now $(\alpha T)(\frac{1}{\alpha}T^{-1}) = \alpha(\frac{1}{\alpha})T(T^{-1}) = 1I = I$

$\Rightarrow (\alpha T)^{-1} = \frac{1}{\alpha}T^{-1}$.

(iii) Let $y_1, y_2 \in V$. *Then there exists* $x_1, x_2 \in V$ *such that* $y_1 = T(x_1)$ *and* $y_2 = T(x_2)$.

$\Rightarrow T^{-1}(y_1) = x_1, T^{-1}(y_2) = x_2$.

Now Suppose that $T^{-1}(y_1) = T^{-1}(y_2)$

$\Rightarrow x_1 = x_2$

$\Rightarrow T(x_1) = T(x_2)$

$\Rightarrow y_1 = y_2$.

This shows that T^{-1} *is one-one.*

Now, since T *is onto, so for each* $y \in V$ *there exists* $x \in V$ *such that* $T(x) = y$.

$\Rightarrow$ *for each* $x \in V$ *there exists* $y \in V$ *such that* $x = T^{-1}(y)$ *[because* T *is invertible and* T^{-1} *is a function]. Therefore* T^{-1} *is also onto. Hence* T^{-1} *is* *one-one and onto* $\Rightarrow T^{-1}$ *is invertible operator on* V.

Also, we have $T^{-1}T = TT^{-1} = I$

$\Rightarrow (T^{-1})^{-1} = T$.

(12.4.6) Example: *Let* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *be a linear transformation defined by*

$$T(x,\, y,\, z) = (3x,\, x - y,\, 2x + y + z).$$

Prove that T *is invertible and find* T^{-1}.

Solution *We know that* T *is invertible if and only if* T *is one-one and onto.*

(i) T *is one-one:*

$Ker(T) = \{(x,\, y,\, z)|T(x,\, y,\, z) = (0,\, 0,\, 0)\}$

$= \{(x,\, y,\, z)|(3x,\, x - y,\, 2x + y + z) = (0,\, 0,\, 0)\}$

$= \{(x,\, y,\, z)|3x = 0,\, x - y = 0,\, 2x + y + z = 0\}$

$= \{(x,\, y,\, z)|x = 0,\, y = 0,\, z = 0\}$

$= \{(0,\, 0,\, 0)\}$

$\Rightarrow Ker(T) = \{(0,\, 0,\, 0)\}$. *Therefore,* T *is one-one.*

(ii) T *is onto: Let* $(a,\, b,\, c) \in \mathbb{R}^3$ *be any element and suppose there exists* $(x,\, y,\, z) \in \mathbb{R}^3$ *such that* $T(x,\, y,\, z) = (a,\, b,\, c)$

$\Rightarrow (3x,\, x - y,\, 2x + y + z) = (a,\, b,\, c)$

$\Rightarrow 3x = a,\, x - y = b,\, 2x + y + z = c$

$\Rightarrow x = \frac{a}{3},\, y = x - b,\, z = c - 2x - y$

$\Rightarrow x = \frac{a}{3},\, y = \frac{a}{3} - b,\, z = c - 2(\frac{a}{3}) - (\frac{a}{3} - b)$

$\Rightarrow x = \frac{a}{3},\, y = \frac{a}{3} - b,\, z = c - a + b.$

Therefore, there exists $(\frac{a}{3},\, \frac{a}{3} - b,\, -a + b + c) \in \mathbb{R}^3$ *such that*

$T(\frac{a}{3},\, \frac{a}{3} - b,\, -a + b + c) = (a,\, b,\, c)$. *Thus* T *is onto.*

Hence T *is one-one and onto* $\Rightarrow T$ *is invertible.*

We have $T(x,\, y,\, z) = (a,\, b,\, c)$

$\Rightarrow T^{-1}(a,\, b,\, c) = (x,\, y,\, z) = (\frac{a}{3},\, \frac{a}{3} - b,\, -a + b + c)$

$\Rightarrow T^{-1}(x, y, z) = (\frac{x}{3}, \frac{x}{3} - y, -x + y + z)$ *is the required inverse of* T.

(12.5) Let Us Sum Up: *As we have seen in set theory, every bijective map is invertible and inverse is also bijective. With the same curiuosity, we have seen in this lesson that bijective linear transformation is invertible. Moreover, the injective linear transformation on finite dimensional vector spaces is also invertible and vice-a-versa. We have explicitly computed the inverse of invertible linear transformation on finite dimensional vector space.*

(12.6) Lesson End Exercise

1. *Let* T *be a linear operator on* $\mathbb{R}^3$ *defined by*
$T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$. *Show that* T *is invertible and find* T^{-1}.
2. *Let* T *be a linear operator on* $\mathbb{R}^3$ *defined by*
$T(x, y, z) = (x - 2y - z, y - z, z)$. *Show that* T *is invertible and find* T^{-1}.
3. *Show that each of the following linear operators* T *is invertible and find the formula for* T^{-1}
(*i*) $T(x, y, z) = (x - 3y - 2z, y - 4z, x)$
(*ii*) $T(x, y, z) = (x + z, x - z, y)$.
4. *Let* T *be a linear operator on* $\mathbb{R}^3$ *defined by*
$T(x, y, z) = (x - 3y - 2z, x - 4z, z)$. *Show that* T *is invertible and find* T^{-1}.
5. *If* T *is a linear transformation on* $T(x, y) = (\alpha x + \beta y, ax + by)$ *for* $(x, y) \in \mathbb{C}^2$ *and* $\alpha, \beta, a, b \in \mathbb{C}$. *Prove that* T *is invertible if and only if* $b\alpha - a\beta \neq 0$.

Hint T *is invertible if and only if* T *is one-one and onto.*
T *is invertible if and only if* $Ker(T) = \{(0, 0)\}$ *i.e.* $Ker(T) = \{(0, 0)\}$
$\{(x, y) | T(x, y) = (0, 0)\} = \{(0, 0)\}$
$\{(x, y) | (\alpha x + \beta y, ax + by) = (0, 0)\} = \{(0, 0)\}$

$\{(x,\, y) | \alpha x + \beta y = 0,\, ax + by = 0\} = \{(0,\, 0)\}$

$\Leftrightarrow \begin{vmatrix} \alpha & \beta \\ a & b \end{vmatrix} \neq 0 \Leftrightarrow b\alpha - a\beta \neq 0$

(12.7) University Model Questions

1. *Let S, T be linear operators on a vector space $V(F)$. Show that T and S are invertible if and only if TS and ST are invertible.*

2. *Let V and W be vector spaces over the same field F such that* $\dim(V) = \dim(W)$ *and $T : V \to W$ is linear transformation. Then prove that T is invertible if and only if T is non-singular.*

(12.8) Suggested Readings *:(i) N.S. Gopalakrishnan, University Algebra, New Age International (P) Limited, Publishers.*

(ii) Kenneth Hoffman, Ray Kunze, Linear Algebra,Prentice Hall India.

Unit-IV

Lesson-XIII **Open and Closed set of $\mathbb{R}$**

(13.1) Introduction

In the previous lesson, we have seen that the denumerable sets are "small" whereas the non-denumberable sets are big. In this lesson, we will see that some sets are "thick" that is they contain an entire neighbourhood of each of its points. We shall be dealing only with real numbers and sets of real numbers unless otherwise stated.

(13.2) Objective

In this lesson, we shall study the concept of neighbourhood of a point, open sets and closed sets on the real line , their examples and properties.

(13.3) Open Sets

(13.3.1) Definition *A set $N \subseteq \mathbb{R}$ is called the neighbourhood of a point a, if there exists an open interval I containing a and contained in N, i.e.,*

$$a \in I \subseteq N$$

Remark:

1. *Every open interval is a neighbourhood of each of its points.*

2. *The set $\mathbb{R}$ is the neighbourhood of each of its points.*

3. *The closed interval [a,b] is the neighbourhood of each point of (a,b) but it is not the nbd. of the end points a and b.*

4. *The empty set is nbd. of each of its points in the sense that there is no point in empty set of which it is not a nbd.*

5. *A non-empty finite set is not a nbd. of any of its points.For, a set can be a nbd. of a point if it contains an interval containing that point. Since an interval necessarily contains an infinite number of points, therefore in order that a set be a nbd. of a point it must necessarily contain an infinty of points.*

6. *The set $\mathbb{Q}$ of rationals, the set $\mathbb{Z}$ of integers, the set $\mathbb{N}$ of natural numbers are not the nbd of any of their points.*

(13.3.2) Definition *Let $A \subseteq \mathbb{R}$. Then A is said to be open if it is a nbd. of each of its points. Equivalently, A is open if for each $x \in A, \exists \epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset A$.*
In the light of the above remark and the definition of the open set, it is clear that:

1. *Every open interval is an open set.*

2. *The set $\mathbb{R}$ is open.*

3. *The closed interval [a,b] is not an open set as it is not the nbd. of the end points a and b.*
4. *The empty set is an open set .*
5. *A non-empty finite set is not an open set.*
6. *The set $\mathbb{Q}$ of rationals, the set $\mathbb{Z}$ of integers, the set $\mathbb{N}$ of natural numbers are not open sets.*
7. *The set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is not open.*

(13.4) Properties of Open sets

(13.4.1) Theorem *Any union of open sets is open.*

Proof. *Let $\{A_\lambda : \lambda \in \Delta\}$ be the family of open sets. We shall show that $\bigcup_{\lambda\in\Delta} A_\lambda$is open. For this, let $x \in \bigcup_{\lambda\in\Delta} A_\lambda$. Then $x \in A_\lambda, for\ some\ \lambda \in \Delta$. Since each A_λ is open, there exists some $\epsilon > 0$ such that $x \in (x-\epsilon, x+\epsilon) \subseteq A_\lambda$. Thus, $x \in (x-\epsilon, x+\epsilon) \subseteq \bigcup_{\lambda\in\Delta} A_\lambda$. This proves that $\bigcup_{\lambda\in\Delta} A_\lambda$is open. Hence the proof.*

(13.4.2) Theorem *Finite intersection of open sets is open.*

Proof. *Let A and B be any two open sets. We shall show that $A \cap B$ is an open set. For this, let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since A and B are open sets, there exist $\epsilon_1 > 0 and \epsilon_2 > 0$ such that $x \in (x-\epsilon_1, x+\epsilon_2) \subseteq A$ and $x \in (x-\epsilon_2, x+\epsilon_2) \subseteq B$. Let $\epsilon =\ min\ \{\epsilon_1, \epsilon_2\}$. Clearly, $x \in (x-\epsilon, x+\epsilon) \subseteq A \cap B$. This proves that $A \cap B$ is open. Hence the proof.*

(13.4.3) Theorem *Prove that every open set is a union of open intervals.*

Proof. *Let A be an open set and $x_\lambda \in A$. Since A is open, there is an open interval I_{x_λ} for each of its points x_λ such that*

$$x_\lambda \in I_{x_\lambda} \subseteq A, \qquad \forall x_\lambda \in A.$$

Again the set A can be thought as the union of singleton sets like $\{x_\lambda\}$. Therefore,

$$A = \bigcup \{x_\lambda\} \subseteq \bigcup I_{x_\lambda} \subseteq A$$

implies,

$$A = \bigcup I_{x_\lambda}$$

. Hence the proof.

(13.4.5) Definition *A point x is said to be an **interior point** of a set S if S is a nbd. of x. The collection of all interior points of a set is called the **interior** of the set. The interior of a set S is generally denoted by S^o.*

(13.4.6)Theorem *Interior of a set is an open set.*

Proof. *Let S be a given set and S^o be its interior.*

If $S^o = \phi$, then S^o is open. Let $S^o \neq \phi$ and let $x \in S^o$. Then x is an interior point of S, there exist an open interval I_x such that $x \in I_x \subseteq S$. But I_x being an open interval, is a nbd. of each of its points. This implies, every point of I_x is an interior point of I_x and $I_x \subseteq S$. Therefore, every point of I_x is an interior point of S. This implies, $I_x \subseteq S^o$. That is, $x \in I_x \subseteq S^o$. This implies that every point of S^o is an interior point of S^o. Hence S^o is an open set.

(13.4.7)Theorem *The interior of a set S is the largest open subset of S.*

Proof *We already know that interior of a set S is an open subset of S. We shall now show that any open subset A of S is contained in S^o. For this, let x be any point of A. Since an open set is nbd. of each of its points, therefore A is a nbd of x. But S is a superset of A, it follows that S is also a nbd. of x. This implies, x is an interior point of S and therefore $x \in S^o$.*

That is, $x \in A$ implies $x \in S^o$.

Therefore, $A \subseteq S^o$.

Hence, every open subset of S is contained in S^o. Thus, the interior of S is *the largest open subset of S.*

Observation:

1. *Any intersection of open sets need not be open. Let* $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right), n \in \mathbb{N}$. *Then* $\{I_n\}_{n\in\mathbb{N}}$ *is an infinite family of open sets and* $\bigcap I_n = \{0\}$, *which being a non-empty finite set is not an open set.*

2. *Every open interval is an open set. But every open set need not be an open interval, for* $A = (0,1) \cup (3,4)$ *is an open set being the union of two open sets but A is not an interval.*

3. *Every open set is a union of open intervals. Lets S be an open set and* $x_\lambda \in S$. *Then there exist an open interval say,* I_{x_λ} *for each* $x_\lambda \in S$ *such that*
$$x_\lambda \in I_{x_\lambda} \subseteq S, \forall x_\lambda \in S.$$
Clearly, $S = \bigcup I_{x_\lambda}$.

(13.5) Closed Sets

(13.5.1) Definition *Let A be a subset of* $\mathbb{R}$. *Then a is said to be closed if its compliment* $\mathbb{R} \setminus A$ *is an open set.*

Remark:

1. *Every closed interval [a,b] is a closed set as* $\mathbb{R} \setminus [a,b] = (-\infty, a) \cup (b, \infty)$ *is an open set.*

2. *The set* $\mathbb{R}$ *is closed as* $\mathbb{R} \setminus \mathbb{R} = \phi$, *is an open set.*

3. *The empty set is closed as* $\mathbb{R} \setminus \phi = \mathbb{R}$, *is an open set.*

4. *The sets (a,b] and [a,b) are neither open nor closed sets.*

(13.5.2) Theorem *Arbitrary intersection of closed sets is closed.*

Proof. *Let $\{A_\lambda : \lambda \in \Delta\}$ be the family of closed sets. We shall show that $\bigcap_{\lambda\in\Delta} A_\lambda$is closed. For this, we shall show that $\mathbb{R} \setminus \bigcap_{\lambda\in\Delta} A_\lambda$ is an open set. Cleraly,*

$$\mathbb{R} \setminus \bigcap_{\lambda\in\Delta} A_\lambda = \bigcup (\mathbb{R} \setminus A_\lambda)$$

Since each A_λ is a closed set, it follows that $\mathbb{R} \setminus A_\lambda$ is open for all λ. Also, arbitrary union of open sets is open, so $\bigcup (\mathbb{R} \setminus A_\lambda)$ is open. Thus, $\mathbb{R} \setminus \bigcap_{\lambda\in\Delta} A_\lambda$ is an open set. Hence the proof.

(13.5.3) Theorem *Finite union of closed sets is closed.*

Proof.: *Let A and B be any two closed sets. To show that $A \cup B$ is closed , we shall show that $\mathbb{R} \setminus (A \cup B) = \mathbb{R} \setminus A \cap \mathbb{R} \setminus B$ is open. Now, A and B are closed implies $\mathbb{R} \setminus A$ and $\mathbb{R} \setminus B$ are open. Since finite intersection of open sets is open, we have $\mathbb{R} \setminus A \cap \mathbb{R} \setminus B$ is open. Hence the proof.*

Observation:

1. *Any union of closed sets need not be closed. Let $A_n = \left[\frac{1}{n}, 1\right], n \in \mathbb{N}$. Then $\{A_n\}_{n\in\mathbb{N}}$ is an infinite family of closed sets and $\bigcup A_n = (0,1]$, which is not a closed set.*

2. *The set of real numbers is a closed set as its complement is empty set, which is open.*

3. *The set of integers is not a closed set.*

4. *The set of rational numbers is not a closed set.*

(13.6) Let us sum up: *In this lesson, we have defined open sets and closed sets on real line. Open intervals are the open sets on real line. We also studied that arbitrary union of open sets is open. The set of real numbers and the empty set are both open and closed.Also, there are sets which are neither open nor closed.*

(13.7) Lesson End Exercise

a. *Give an example of each of the following:*

1 *a set which is a nbd. of each of its points.*

Sol. *The open interval* $(1,2)$.

2 *a set which is not nbd. of any of its points.*

Sol. *The finite set* $\{1,2,3,4,5\}$.

3 *a set which is a nbd. of each of its points with the exception of one point.*

Sol. *The set* $(1,4]$ *is nbd. of each of points except 4.*

4 *a set which is a nbd. of each of its points with the exception of two points.*

Sol. *The closed interval* $[5,7]$ *is nbd. of each of its points except the end points 5 and 7.*

5 *a set which is a nbd. of each of its points with the exception of n points,* $n \geq 1$.

Sol. *The set* $S = (0,1) \bigcup \{1,2,3,4,5...,n\}$, *is a nbd. of each of its points except n points 1,2,3,4,5,....,n.*

b. *Give an example of each of the following:*

1 *an open set which is not an interval.*

Sol. *The set* A=$(1,2)\bigcup(3,4)$. *is an open set being the finite union of two open sets. But* A *is not an interval.*

2 *an interval which is not an open set.*

Sol. $[2,3]$ *is an interval but not an open set.*

3 *a set which is neither an interval nor an open set.*

Sol. *The finite set* $\{1,2,3\}$ *is neither an interval nor an open set.*

c. *Which of the following are closed,open, neither open nor closed set?*

1 $\{x : 0 \leq x \leq 1\}$

2 $[0,1] \cup [2,3]$

3 $\{x : 1 < \ x \ < 7\}$

4 $\{x : 4 \leq x < 6\}$

5 *The set of integers* $\mathbb{Z}$

6 *The set of rationals* $\mathbb{Q}$

Sol c.

1 $\{x : 0 \leq x \leq 1\} = [0,1]$, *being a closed interval is a closed set.*

2 $[0,1] \cup [2,3]$, *is a closed set being the finite union of closed sets* $[0,1]$ *and* $[2,3]$.

3 $\{x : 1 < \ x \ < 7\} = (1,7)$, *being an open interval is an open set.*

4 $\{x : 4 \leq x < 6\} = [4,6)$, *is neither open nor closed.*

5 *The set of integers* $\mathbb{Z}$ *is not closed as it is not the nbd. of any of its points. The set of integers has no limit points and therefore* $\mathbb{Z}$ *is a closed set.*

6 *The set of rational numbers* $\mathbb{Q}$ *is not the nbd. of any of its points. Also, it doesnot contain all of its limit points, therefore the set of rational numbers is neither open nor closed.*

d. *Prove that* $\mathbb{R} - \mathbb{N}$ *and* $\mathbb{R} - \mathbb{N}$ *are open sets.*

Sol. *Let* $x \in \mathbb{R} - \mathbb{N} = \mathbb{N}^c$, *then* $x \notin \mathbb{N}$, *that is x is not a natural number. If n is the natural number nearest to x, then there exists* $\epsilon = \frac{|x-n|}{2} > 0$ *s.t* $(x-\epsilon, x+\epsilon)$ *does not contain any natural number, i.e.,* $(x-\epsilon, x+\epsilon) \cap \mathbb{N} = \phi$

Therefore, $(x-\epsilon, x+\epsilon) \subset \mathbb{N}^c$

$\Rightarrow \quad \mathbb{N}^c$ *is a nbd of x.*

$\Rightarrow \quad \mathbb{N}^c$ *is open.*

Hence, $\mathbb{R} - \mathbb{N}$ *is open.*

Again, let $x \in \mathbb{R} - \mathbb{Z} = \mathbb{Z}^c$, *then* $x \notin \mathbb{Z}$, *that is x is not an integer.*

If n is the integer nearest to x, then there exists $\epsilon = \frac{|x-n|}{2} > 0$ *s.t*

$(x-\epsilon, x+\epsilon)$ *does not contain any point of* $\mathbb{Z}$, *i.e.,* $(x-\epsilon, x+\epsilon) \cap \mathbb{Z} = \phi$

Therefore, $(x-\epsilon, x+\epsilon) \subset \mathbb{Z}^c$

$\Rightarrow \quad \mathbb{Z}^c$ *is a nbd of* x.

$\Rightarrow \quad \mathbb{Z}^c$ *is open.*

Hence, $\mathbb{R} - \mathbb{Z}$ *is open.*

(13.8) University Model Questions

1. *Define open sets. Give two examples. Show that the arbitrary union of open sets is open.*

2. *Show that every finite set is closed.*

3. *Show that every non-empty open set is a union of open intervals.*

4. *Give an example to show that*

   i *a subset of a closed set need not be closed.*

   ii *a set containing a closed set need not be closed.*

5. *Let A be a closed set and B be an open set. Show that B-A is an open set.*

(13.9) Suggested Readings

1 *T. M. Apostol, Calculus (Vol. I), John Wiley and Sons (Asia) P. Ltd., 2002*

2. *S. C. Malik and S. Arora, Mathematical Analysis, New Age international Publishers, 2010.*

3. *R.G. Bartle and D. R Sherbert, Introduction to Real Analysis, John Wiley and Sons (Asia) P. Ltd.,2000.*

Lesson-XIV Denumerable and Non-Denumerable Sets

(14.1) Introduction: *With the notion of bijection, it is easy to formalize the idea that two finite sets have same number of elements. We just need to verify that their elements can be placed in pairwise correspondence. It is natural to generalize this to infinite sets and indeed to any arbitrary sets.*

(14.2) Objective

One is led to consider some unusual subsets of the real line and it is natural to wonder if one can give a precise intuitive meaning to the feeling that some infinite sets have more elements than other infinite sets.(for example, real line seems to have more elements than the natural numbers in it.)

(14.3) Denumerable Sets

The notion of equivalence of sets is supposed to lead us to a notion of relative sizes of sets. Equivalent sets should by rights have same number of elements.

(14.3.1) Definition. *Two sets X and Y are said to be* **equivalent**, *symbolized by $X \sim Y$, if there exists a one to one correspondence $f : X \longrightarrow Y$.*

Remark:

1. *Equivalence is an equivalence relation on class of sets.*

2. *Any two open(closed) intervals are equivalent.*

3. *Any open interval is equivalent to the set of real numbers.*

4. *If X,Y,Z and W are sets with* $X \cap Z = \phi = Y \cap W$ *and* $X \sim Y$ *and* $Z \sim W$*, then* $(X \cup Z) \sim (Y \cup W)$.

5. *If X,Y,Z and W are sets such that* $X \sim Y$ *and* $Z \sim W$*, then* $(X \times Z) \sim (Y \times W)$.

(14.3.2) Definition *A set X is said to be* **finite** *if it is either empty or* $X \sim \mathbb{N}_k$ *, where* $\mathbb{N}_k = \{1, 2, 3, 4, 5, ..., k\}$.

(14.3.3) Definition *A set X is said to be* **denumerable** *provided that* $X \sim \mathbb{N}$.

Remarks:

1. *A denumerable set can be thought of as the smallest infinte set.*

2. *Let X be a denumerable set. then there is a bijection* $f : \mathbb{N} \longrightarrow X$*. If we denote*
   $f(1) = x_1, f(2) = x_2, ...f(k) = x_k, ...$*, then the elements of X be put in a sequence* $\{x_1, x_2, ..., x_k, ...\}$.

3. *Every infinite subset of a denumerable set is denumerable.*

4. *If X is a denumerable set and Y is a finite set then $X \cup Y$ is denumerable.*

(14.4) Examples and Properties of Denumerable Sets

(14.4.1) Theorem. *The set of all even natural numbers $\mathbb{N}_e = \{2n : \ n \in \mathbb{N}\}$ is denumerable.*

Proof *To show that the set of even natural numbers is denumerable, consider the mapping $f : \mathbb{N} \longrightarrow \mathbb{N}_e$ defined as,*

$$f(n) = 2n$$

f is one – one

Let $n, m \in \mathbb{N}$ be such that

$$f(n) = f(m)$$

implies,

$$2n = 2m$$

implies,

$$n = m$$

f is onto

Clearly, for each $y=2n \in \mathbb{N}_e$ there is $n \in \mathbb{N}$ such that $f(n)=2n=y$.

Therefore, f is a bijection. Hence, the set of even natural numbers is denumerable.

(14.4.2) Theorem *The set of integers,* $\mathbb{Z}$ *is denumerable.*

Proof. *Consider the function,* $f : \mathbb{N} \longrightarrow \mathbb{Z}$, *defined as*

$$f_n = \begin{cases} \frac{n}{2}\ ; & \text{if } n \text{ is even,} \\ -\frac{n-1}{2}\ ; & \text{if } n \text{ is odd.} \end{cases}$$

We shall show that f *is a bijective map.*

f is one – one

Let $n,m \in \mathbb{N}$ *be such that* $f(n) = f(m)$. *Then,*

Case1: *When both* n *and* m *are even natural numbers. Now,*

$$f(n) = f(m)$$

implies,

$$\frac{n}{2} = \frac{m}{2}$$

implies,

$$n = m.$$

Case2:*When both* n *and* m *are odd natural numbers. Now,*

$$f(n) = f(m)$$

implies,

$$-\frac{n-1}{2} = -\frac{m-1}{2}$$

implies,

$$n - 1 = m - 1.$$

implies,

$$n = m.$$

f is onto

Let $y \in \mathbb{Z}$. ***Case1***: y *is a positive integer.*

Subcase1:*y is a positive even integer.*

Let y=2n, for some natural number n. Then there is some x=4n , n$\in \mathbb{N}$ *such that* $f(x) = \frac{4n}{2} = 2n = y$.

Subcase2:*y is a positive odd integer.*

Let y=2n-1, for some natural number n. Then there is some x=4n-2 , n$\in \mathbb{N}$ *such that* $f(x) = \frac{4n-2}{2} = 2n - 1 = y$.

Case2: *y is a negative integer.*

Subcase1:*y is a negative even integer.*

Let y=-2n, for some natural number n. Then there is some x=4n+1 , n$\in \mathbb{N}$ *such that* $f(x) = -\frac{(4n+1)-1}{2} = -2n = y$.

Subcase2:*y is a negative odd integer.*

Let y=-$(2n-1)$*, for some natural number n. Then there is some x=4n-1 , n*$\in \mathbb{N}$ *such that* $f(x) = -\frac{(4n-1)-1}{2} = -(2n-1) = y$.

Clearly, pre image of 0$\in \mathbb{Z}$ *is 1. Therefore, f is onto.*

Hence, f is a bijective map. This proves that $\mathbb{Z}$ *is denumerable.*

(14.4.3) Theorem *The union of two denumerable sets is denumerable.*

Proof. *Let A and B be any two denumerable sets. We shall show that* $A \cup B$ *is denumerable.*

Case 1: $A \cap B = \phi$

Since $A \sim \mathbb{N}$ *and* $\mathbb{N} \sim \mathbb{N}_o$*, we have* $A \sim \mathbb{N}_o$*. Similarly, we have* $B \sim \mathbb{N}_e$*. Consequently, we have* $(A \cup B) \sim (\mathbb{N}_o \cup \mathbb{N}_e) = \mathbb{N}$*, which shows that* $A \cup B$ *is denumerable.*

Case 2: $A \cap B \neq \phi$

Let $C = B \setminus A$*. Then* $A \cup C = A \cup B$ *and* $A \cap C = \phi$*. Also,* $C \subseteq B$ *is either finite or denumerable. If C is finite, then* $A \cup C$ *is denumerable as union of a finite set with a denumerable set is denumerable. If C is denumerable, then by* *case 1 $A \cup C$ is denumerable. Hence, the set $A \cup B$ is denumerable.*

Remark:

1. *Finite union of denumerable sets is denumerable.*

2. *The set $\mathbb{Z}$ is denumerable as $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup -\mathbb{N}$.*

(14.4.4) Theorem *The set $\mathbb{N} \times \mathbb{N}$ is denumerable.*

Proof *: Consider the function $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ given by $f(j,k) = 2^j 3^k$, for all $(j,k) \in \mathbb{N} \times \mathbb{N}$. this function is injective, so that $\mathbb{N} \times \mathbb{N} \sim f(\mathbb{N} \times \mathbb{N}) \subset \mathbb{N}$. Since $\mathbb{N} \times \mathbb{N}$ is infinite, so is $f(\mathbb{N} \times \mathbb{N})$. Since infinite subset of a denumerable set is denumerable, it follows that $f(\mathbb{N} \times \mathbb{N})$ is denumerable and so is $\mathbb{N} \times \mathbb{N}$. Hence the proof.*

(14.4.4) Theorem. *The set of rational numbers $\mathbb{Q}$ is denumerable.*

Proof*: We can represent each rational number uniquely as $\frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ and the greatest common divisor of p and q is 1. Let $\mathbb{Q}_+$ be the set of all such $\frac{p}{q} > 0$ and let $\mathbb{Q}_-$ be the set of all such $\frac{p}{q} < 0$. Then $\mathbb{Q} = \mathbb{Q}_+ \cup \{0\} \mathbb{Q}_-$. Clearly, $\mathbb{Q}_+ \sim \mathbb{Q}_-$. hence, to show that $\mathbb{Q}$ is denumerable, it is sufficient to show that $\mathbb{Q}_+$ is denumerable. For this, we consider a function $f : \mathbb{Q}_+ \longrightarrow \mathbb{N} \times \mathbb{N}$ given by $f\left(\frac{p}{q}\right) = (p,q)$. Since f is injective , we have $\mathbb{Q}_+ \sim f(\mathbb{Q}_+) \subseteq \mathbb{N} \times \mathbb{N}$. Also, $\mathbb{Q}_+$, is infinite so $f(\mathbb{Q}_+)$ is an infinite subset of the denumerable set $\mathbb{N} \times \mathbb{N}$. Therefore, $f(\mathbb{Q}_+)$ is denumerable and consequently$\mathbb{Q}_+$ is denumerable. Hence the proof.*

(14.5) Non-Denumerable Sets

(14.5.1) Definition *A set X is said to be non-denumerable if it is not denumerable.*

(14.5.2) Theorem. *The open unit interval $(0,1)$ of real numbers is a non-*

denumerable set.

Proof *: Each* $x\in (0,1)$, *can be expressed in the form* $.x_1x2x_3...$, *where each* $x_i \in \{0,1,2,3,4...,9\}$ *for all* $n \in \mathbb{N}$. *For example,* $\frac{1}{3} = .333333...$.*In order to have a unique infinite decimal expression , for those numbers with a terminating decimal expansion such* $\frac{1}{4} = .25$, *we append 9's so that* $\frac{1}{4} = .2499999...$ *and not as* $\frac{1}{4} = .250000...$

Now suppose that the set $(0,1)$ *is denumerable. Then there exists a bijection* $f : \mathbb{N} \longrightarrow (0,1)$. *So, we may list all elements of* $(0,1)$ *as follows:*

$$\begin{aligned}
f(1) &= .a_{11}a_{12}a_{13}... \\
f(2) &= .a_{21}a_{22}a_{23}... \\
f(3) &= .a_{31}a_{32}a_{33}... \\
&. \\
&. \\
&. \\
f(k) &= .a_{k1}a_{k2}a_{k3}... \\
&. \\
&. \\
&.
\end{aligned}$$

, *where each* $a_{i,j} \in \{0,1,2,3,4,...,9\}$. *Let* $z = z_1z_2z_3...$ *be defined by* $z_k = 5if\ a_{kk} \neq 5$ *and* $z_k = 1\ if a_{kk} = 5$, *for each* $k \in \mathbb{N}$. *Clearly,* $z \in (0,1)$ *but* $z \neq f(k)$, *for any* $k \in \mathbb{N}$, *which is a contradiction. Thus,* $(0,1)$ *is non-denumberable.*

Observation:

1. *Since* $(0,1) \subset \mathbb{R}$, *it follows that the set of real numbers is non-denumerable.*

2. *The set of irrational numbers is non denumerable. For, if* $\mathbb{R} \setminus \mathbb{Q}$ *is denumerable, then the union* $(\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = \mathbb{R}$ *is denumerable, which is a contradiction.*

(14.6)Let us sum up: *In this lesson, we defined denumerable and non-denumerable sets. Denumerable sets are considered as "'small"' infinite sets, while non- denumerable sets are considered as "'big"' infinite sets. From this point of view, the set of natural numbers, the set of integers and the the set of rational numbers are all small relative to the set of real numbers.*

(14.7) Lesson End Exercise

1. *Prove that the set of all sequences whose elements are either zero or one is not countable.*

2. *Prove that the set* A=2^m : m *is an integer. is countable.*
   Hint *Define a mapping* $f : \mathbb{Z} \longrightarrow A$ *as* $f(m) = 2^m$. *Use the fact that the set of integers is equivalent to the set of naturals and equivalence of sets is a transitive relation.*

3. *Show that the set of prime numbers is denumerable.*
   Sol *As the set of natural numbers is denumerable sets and the set of prime numbers is a subset of natural numberss.Also the set of prime numbers is an infinite subset of the denumerable set* $\mathbb{N}$, *and we have that*

an infinite subset of a denumerable set is denumerable, it follows that the set of prime numbers is denumerable.

4. *Show that if A and B are denumerable sets then $A\times$ B is also a denumerable set.*

   Hint *Since A and B are denumerable sets, there exists bijections $f : \mathbb{N} \longrightarrow A$ and $g : \mathbb{N} \longrightarrow B$. Define a map $h : \mathbb{N} \longrightarrow A \times B$ as $h(n) = (f(n), g(n))$. Clearly, h is a bijection.*

5. *Show that the set of all odd natural numbers,$\mathbb{N}_o$ is denumerable.*

(14.8) University Model Questions

1. *Define Denumerable sets. Give two examples. Show that the set of rational numbers is denumerable.*

2. *Show that the finite union of denumerable sets is denumerable and hence show that the set of irrational numbers is non-denumerable.*

3. *Show that the set of complex numbers is non-denumerable.*

4. *Show that the interval $[0, 1]$ is non-denumerable and hence show that the set of real numbers is non-denumerable.*

5. *Find a bijection between the set of integers and the set of rational numbers.*

(14.9) Suggested Readings

1 *T. M. Apostol, Calculus (Vol. I), John Wiley and Sons (Asia) P. Ltd., 2002*

2. *S. C. Malik and S. Arora, Mathematical Analysis, New Age international Publishers, 2010.*

3. *You-Feng Lin, Shwu Yeng T. Lin, Set Theory With Applications-Mariner Publishing Company (1981).*

(15.1) Introduction:*In this lesson, we are going to study the notion of limit points of a set and some important results based on the concept of limit points of a set in $\mathbb{R}$. The notion of limit point is an extension of the notion of being "close" to a set in the sense that it tries to measure how crowded the set is. To be a limit point of a set, a point must be surrounded by infinitely many points of the set.*

(15.2) Objective

The main objective of this lesson is to make students familiar with the notion of limit points of a set which is fundamental for laying the foundation of real analysis.

(15.3) Limit Points of a Set

(15.3.1) Definition. *A real number l is said to be the* **limit point** *of a set $S \subset \mathbb{R}$, if every neighbourhood N of l contains a point of S other than l. That is,*

$$(N \cap S) \setminus \{l\} \neq \phi.$$

Equivalently, a real number l is said to be the limit point of a set $S \subset \mathbb{R}$, if *every neighbourhood N of l contains infinitely many points of S. Limit point of a set is also called as the accumulation point or the cluster point or the condensation point.*

Observation:

1. *The set of integers has no limit point, for a nbd.* $\left(m - \frac{1}{2}, m + \frac{1}{2}\right)$ *of* $m \in \mathbb{Z}$*, contains no point of* $\mathbb{Z}$ *other than m.*

2. *Every point of* $\mathbb{R}$ *is a limit point, for every nbd. of any of its points contains infinite members of* $\mathbb{R}$.

3. *Every point of the set* $\mathbb{Q}$ *of rationals is a limit point for between any two rationals there are infinite rational numbers. Also every irrational number is also a limit point of* $\mathbb{Q}$ *for between any two irrationals there are infinite rational numbers. Thus, every real number is a limit point of the set of rationals.*

4. *The set* $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ *has only one limit point, zero, which does not belong to the set.*

5. *Every point of the open interval* $\{a, b\}$ *is its limit point. The end points a, b which are not in the set are also its limit points.*

6. *A finite set has no limit point.*

7. *The derived set of the set* $\left\{\frac{1}{m} + \frac{1}{n} :\ m, n \in \mathbb{N}\right\}$ *is* $\{0\} \cup \left\{\frac{1}{n} :\ n \in \mathbb{N}\right\}$

8. *The set* $\left\{1 + \frac{1}{n}, n \in \mathbb{N}\right\}$ *has only one limit point 1.*

9. *1 and -1 are the limit points of the set* $\left\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, ...\right\}$.

(15.3.2) Definition *The set of all limit points of a set S is called the* ***derived set*** *of S and is denoted by $S^{'}$. Thus,*

$$S^{'} = \{x : x \text{ is a limit point of } S\}.$$

(15.4) Important Results

(15.4.1) Theorem *Prove that a real number l is a limit point of a set S iff each nbd. of l contains infinitely many points of S.*

Proof. *Let l be the limit point of S. Then by definition, every nbd. N of l contains a point of S other than l. That is,*

$$N \cap S \setminus \{l\} \neq \phi.$$

Suppose N contains only finitely many points of S. Let

$$N \cap S \setminus l = \{l_1, l_2, ...l_n\}$$

and $\epsilon = min. \{|l - l_1|, |l - l_2|, ... |l - l_n|\} > 0$.

Then $(l - \epsilon, l + \epsilon)$ is a nbd. of l which contains no point of S. That is,

$$(l - \epsilon, l + \epsilon) \cap S \setminus \{l\} = \phi,$$

a contradiction. Thus,N contains infinitely many points of S. this proves the direct part.

Conversely, let $l \in \mathbb{R}$ be such that each nbd. N of l contains infinitely many points of S.

This implies, every nbd. N of l contains a point of S other than l.

That is,

$$(l - \epsilon, l + \epsilon) \cap S \setminus \{l\} \neq \phi.$$

Thus, l is the limit point of S.

Hence the proof.

(15.4.2) Theorem *Prove that a finite set has no limit point.*

Proof . *Let* $A=\{x_1, x_2, ..., x_n\}$ *be a finite subset of* $\mathbb{R}$*. If possible, assume that* A *has a limit point say* x*.*

Now, if we choose $\epsilon = min.\{|x - x_1|, |x - x_2|, ..., |x - x_n|\}$*, then* $(x - \epsilon, x + \epsilon)$ *is a nbd. of* x *which contains no point of* A*, a contradiction. Hence our supposition was wrong. Since* x *is arbitrary, it follows that* A *has no limit point.*

(15.4.3) Theorem. *Prove that 0 is the limit point of set*

$$S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}.$$

Proof. *For each* $\epsilon > 0$*,* $(-\epsilon, \epsilon)$ *is a nbd. of 0. By Archimedean property of reals, for each* $\epsilon > 0, \exists n \in \mathbb{N}$ *such that* $n > \frac{1}{\epsilon}$

$\Rightarrow \quad \frac{1}{n} < \epsilon$

$\Rightarrow \quad -\epsilon < 0 < \frac{1}{n} < \epsilon$

$\Rightarrow \quad \frac{1}{n} \in (-\epsilon, \epsilon)$

Thus every nbd. of 0 contains a point of S, namely $\frac{1}{n}$*.*

$\Rightarrow$ *0 is the limit point of S.*

Uniqueness.

$S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subset (0, 1]$.

We shall show that there is no real number other than 0 which is a limit point of S. Let x *be a non- zero real number. Then the following cases arise:*

C*ase(i)*

If $x < 0$*, then* $(-\infty, 0)$ *is a nbd. of* x *which contains no pint of S.*

$$i.e., (-\infty, 0) \cap S = \phi.$$

Therefore, x *is not a limit point of S.*

***C**ase(ii)*

If $x > 1$, then $(1, \infty)$ is a nbd. of x which does not contain any point of S.

$$i.e., (1, \infty) \cap S = \phi.$$

Therefore, x is not a limit point of S.

***C**ase(iii)*

If $x = 1$, then $\left(\frac{1}{2}, \infty\right)$ is a nbd. of x which doesnot contain any point of S.

$$\left(\frac{1}{2}, \infty\right) \cap S - \{1\} = \phi.$$

Therefore, x is not a limit point of S.

***C**ase(iv)*

If $0 < x < 1$, then $\frac{1}{x} > 0$

Therefore, there exists a unique natural number n such that

$n \leq \frac{1}{x} < n + 1$

$\Rightarrow \quad \frac{1}{n} \geq x > \frac{1}{n+1}$

$\Rightarrow \quad \frac{1}{n+1} < x \leq \frac{1}{n} \frac{1}{n-1}$

$\Rightarrow$ *the nbd. $\left(\frac{1}{n+1}, \frac{1}{n-1}\right)$ of x contains only one point $\frac{1}{n}$of S, i.e., only finite number of points of S.*

Hence 0 is the only limit point of S.

(15.4.4) Theorem. *Prove that for any set A, A' is a closed set.*

Proof . *To prove that A' is a closed set, we shall show that $\left(A'\right)^c$ is an open set. for this, let $x \in \left(A'\right)^c$.*

$\Rightarrow \quad x \notin A'$

$\Rightarrow$ *x is not a limit point of A.*

$\Rightarrow \exists$ *a nbd. $I = (x - \epsilon, x + \epsilon)$ of x such that*

$I \cap A - \{x\} = \phi.$

Let $y \in I$, then I being an open interval is an open set.

$\Rightarrow$ *I is a nbd. of y. Also,* $I \cap A - \{x\} = \phi$.

$\Rightarrow$ *y is not a limit point of A.*

$\Rightarrow y \notin A' \Rightarrow y \in \left(A'\right)^c$.

Now, $y \in I \Rightarrow y \in \left(A'\right)^c$.

Therefore, $I = (x-\epsilon, x+\epsilon) \subset \left(A'\right)^c$.

$\Rightarrow \left(A'\right)^c$ *is a nbd. of x.*

Since x is any element of $\left(A'\right)^c$, *it follows that* $\left(A'\right)^c$ *is nbd. of each of its points. This proves that* $\left(A'\right)^c$ *is open. Hence,* A' *is a closed set.*

(15.4.5) Theorem *If* $A, B \subset \mathbb{R}$, *then*

i $A \subset B \Rightarrow A' \subset B'$

ii $(A \cup B)' = A' \cup B'$

iii $(A \cap B)' \subset A' \cap B'$

proof.

i. *If* $A' = \phi$, *then* $A' \subset B'$, *since empty set is the subset of every set.*
If $A' \neq \phi$, *let* $x \in A'$ *and N be any nbd. of x.*
$\Rightarrow$ *N contains infinitely many points of A.*
$\Rightarrow$ *N contains infinitely many points of B.*
$\Rightarrow$ *x is a limit point of B. That is,* $x \in B'$.
Now $x \in A'$ *implies* $x \in B'$.
Therefore, $A' \subset B'$.

ii. *Since* $A \subset A \cup B$ *and* $A \subset A \cup B$
$\Rightarrow A' \subset (A \cup B)'$ *and* $B' \subset (A \cup B)'$
$\Rightarrow A' \cup B' \subset (A \cup B)'$ *...1*

Now we proceed to show that $(A\cup B)^{'} \subset A^{'}\cup B^{'}$.

If $(A\cup B)^{'} = \phi$, *then* $(A\cup B)^{'} \subset A^{'}\cup B^{'}$.

If $(A\cup B)^{'} \neq \phi$, *let* $x \in (A\cup B)^{'}$.

$\Rightarrow$ *x is a limit point of* $A\cup B$.

$\Rightarrow$ *every nbd. of x contains infinitely many points* $A\cup B$.

$\Rightarrow$ *every nbd. of x contains infinitely many points of A or B.*

$\Rightarrow$ *x is a limit point of A or a limit point of B.*

$\Rightarrow$ $x \in A^{'}$ *or* $x \in B^{'}$

$\Rightarrow$ $x \in A^{'}\cup B^{'}$

Since $x \in (A\cup B)^{'}$, *implies* $x \in A^{'}\cup B^{'}$ *....2*

From 1 and 2 we have

$$(A\cup B)^{'} = A^{'}\cup B^{'}.$$

iii. $A\cap B \subset A$ *implies* $(A\cap B)^{'} \subset A^{'}$

$A\cap B \subset B$ *implies* $(A\cap B)^{'} \subset B^{'}$

Therefore, $(A\cap B)^{'} \subset A^{'}\cap B^{'}$.

Note: $(A\cap B)^{'}$ *and* $A^{'}\cap B^{'}$ *may not be equal. For example, let* $A=(1,2)$, $B=(2,3)$. *Therefore,* $A\cap B = \phi$, *implies* $(A\cap B)^{'} = \phi$.

Also, $A^{'} = [1,2]$, $B^{'} = [2,3]$.

$A^{'}\cap B^{'} = \{2\}$.

Therefore, $(A\cap B)^{'} \neq A^{'}\cap B^{'}$.

(15.4.6) Definition *Let* $S \subset \mathbb{R}$. *Then the closure of S is defined as the set of all those point in* $\mathbb{R}$ *which are either the points of S or the limit point(s) of S. Closure of S is denoted by* $\overline{S}$. *That is,*

$$\overline{S} = S\cup S^{'}.$$

(15.4.7) Theorem *Prove that for any set A, $\overline{A}$ is a closed set.*

Proof *To show that $\overline{A}$ is closed, it is enough to show that $\overline{A}^c$ is open.*

Let x be any element of $\left(\overline{A}\right)^c$

$x \in \left(\overline{A}\right)^c$

$\Rightarrow x \notin \overline{A}$

$\Rightarrow x \notin A \cup A'$

$\Rightarrow x \notin A$ *and* $x \notin A'$

$\Rightarrow \exists$ *a nbd.* $I = (x - \epsilon, x + \epsilon)$ *of x such that* $I \cap A = \phi$

Let $y \in I$, then I being an open interval is an open set.

$\Rightarrow$ *I is a nbd of y. Also,* $I \cap A = \phi$

$\Rightarrow$ *y is not a limit point of A.*

$\Rightarrow y \notin A'$. *Also,*$y \notin A$.

$\Rightarrow y \notin A \cup A'$. *Also,*$y \notin \overline{A}$.

$\Rightarrow y \in \left(\overline{A}\right)^c$

Since $y \in I \Rightarrow y \in \left(\overline{A}\right)^c$.

Therefore, $I = (x - \epsilon, x + \epsilon) \subset \left(\overline{A}\right)^c$.

$\Rightarrow \quad \left(\overline{A}\right)^c$ *is a nbd. of x.* $\Rightarrow \quad \overline{A}^c$ *is an open set.*

$\Rightarrow \quad \overline{A}$ *is a closed set.*

Hence the proof.

(15.4.8) Theorem *Prove that A set is closed iff $A = \overline{A}$.*

That is, A is closed iff A contains all its limit points.

Proof *If $A = \overline{A}$, then A is closed because $\overline{A}$ is closed.*

Conversely, let A be a closed set. We shall show that $A = \overline{A}$. Clearly,

$A \subset \overline{A}$. *(1)*

If $A' = \phi$ *then* $A' \subset A$.

If $A' = \phi$ *then* $A' \subset A$.

If $A^{'} \neq \phi$ then let $x \in A^{'}$.

Suppose $x \notin A$, then $X \in A^c$. Since A is a closed set, A^c is an open set.

Therefore, A^c is the nbd. of x.

Also, $x \in A^{'}$ $\quad \Rightarrow x$ is a limit point of A.

$\Rightarrow$ every nbd. of x contains infinitely many points of A.

$\Rightarrow A^c$ contains infinitely many points of A.

$\Rightarrow A^c \cap A \neq \phi$, a contradiction.

Thus, our supposition was wrong. Therefore, $x \in A$. Since $x \in A^{'}$ implies $x \in A$.

Therefore $A^{'} \subset A$.

Hence, $\overline{A} \subset A \qquad$ (2).

From (1) and (2) we have $A = \overline{A}$.

Hence the proof.

(15.4.9) Theorem *If A and B are subsets of $\mathbb{R}$, then prove that*

i $\overline{A \cup B} = \overline{A} \cup \overline{B}$

ii $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$

Proof *(i)* $\overline{A \cup B} = (A \cup B) \cup (A \cup B)^{'}$

$= (A \cup B) \cup \left(A^{'} \cup B^{'}\right)$

$= A \cup \left(B \cup A^{'}\right) \cup B^{'}$

$= \left(A \cup A^{'}\right) \cup \left(B \cup B^{'}\right)$

$= \overline{A} \cup \overline{B}$

(ii) $A \cap B \subset A$

$\Rightarrow \quad \overline{A \cap B} \subset \overline{A}$

Also, $A \cap B \subset B$

$\Rightarrow \quad \overline{A \cap B} \subset \overline{B}$

Therefore, $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$

Note: *The inclusion cannot be replaced by equality. For example, if* $A = (0,1)$ *and* $B = (1,2)$ *, then* $A \cap B = \phi$*.*

Therefore, $\overline{A \cap B} = \phi$*.*

Also,$\overline{A} = [0,1]$ *and* $\overline{B} = [1,2]$*.*

Now, $\overline{A} \cap \overline{B} = \{1\}$ *.*

Thus, $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$

(15.5) Let Us Sum Up*: A limit point x of a set S is a point which can be "approximated" by the points of the set S in the sense that every neighbourhood of x contains a point of S other than x itself. Limit point of a set is not unique. A set may or may not have a limit point. Limit point of a set may or may not belong to the set.*

(15.6) Lesson End Exercise

1. *Give an example of an infinite set with no limit point.*

   Sol: *The set of natural numbers,* $\mathbb{N}$*.*

2. *Give an example of a set with exactly one limit point.*

   Sol. *Th set S=*$\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ *has exactly one limit point 0, which dosnot belong to the set S.*

3. *Give an example of a set with exactly two limit points.*

   Sol. *Consider the set S=* $\left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{2 + \frac{1}{n} : n \in \mathbb{N}\right\}$ *has two limit points 1 and 2.*

4. *Give an example of set with infinitely many limit points.*

   Sol. *each point of the set of real numbers is a limit point of* $\mathbb{R}$.

5. *Show that a set closed if it contains all its limit points.*

   Sol. *Let S be a set. Assume that S is not closed. Then* S^c *is not open. Then there is some* $x \in S^c$ *such that some nbd. of x contains a point of S. Clearly, x is a limit point of S which lies in* S^c.

   Thus, S is not closed implies there is a limit point of S which is not in S. Hence a set is closed if it contains all its limit points.

(15.7) University Model Questions

1. *In each situation below, give an example of a set which satisfies the given condition.*

   a. *A bounded set with no limit point.*

   b. *An unbounded set with no limit point.*

   c. *An unbounded set with exactly five limit points.*

   d. *A set whose derived set is whole of real line.*

2. *Define derived set. Show that the derived set of a set is a closed set.*

3. *Show that if x has a nbd. which contains only finitely many members of a set S, then x cannot be a limit point of S.*

4. *Is it true that if A and B are subsets of* $\mathbb{R}$ *then* $(A \cap B)' = A' \cap B'$*? Justify.*

5. *Prove that a finite set has no limit points.*

(15.8) Suggested Readings

1 *T. M. Apostol, Calculus (Vol. I), John Wiley and Sons (Asia) P. Ltd., 2002*

2. *S. C. Malik and S. Arora, Mathematical Analysis, New Age international Publishers, 2010.*

3. *R.G. Bartle and D. R Sherbert, Introduction to Real Analysis, John Wiley and Sons (Asia) P. Ltd., 2000.*

Lesson-XVI

Heine Borel theorem for closed and bounded intervals

(16.1) Introduction:*The notion of compact sets is of prime importance in real analysis. The concept of compactnes is the abstraction of an important property known as 'Heine- Borel Property' posed by subsets of* $\mathbb{R}$ *which are closed and bounded. Heine Borel theorem states that if* $I \subset \mathbb{R}$ *is a closed interval, then any family of open interval in* $\mathbb{R}$ *whose union contains* I *has a finite subfamily which covers* I*. Compactness is concerned with covering sets with open sets.*

(16.2) Objectives: *This lesson aims at studying open cover, compact sets, Heine-Borel Property, Heine-Borel Theorem and some exercises based on these concepts.*

(16.3) Open Covering

(16.3.1) Definition *Let* A *be a non-empty subset of* $\mathbb{R}$*. A family* $\{A_\lambda\}_{\lambda\in\Lambda}$ *of subsets of* $\mathbb{R}$ *is said to be a* **cover** *of* A *if*

$$S \subset \bigcup_{\lambda\in\Lambda} A_\lambda.$$

If each member of $\{A_\lambda\}_{\lambda\in\Delta}$ *is an open set, then the cover is called an open* *cover.*

(16.3.2) Definition *Let A be a non-empty subset of $\mathbb{R}$. and $\{A_\lambda\}_{\lambda\in\Lambda}$ be an open cover of A. If there exists a subset $\Lambda' \subset \Lambda$ such that the sub family $\{A_\lambda\}_{\lambda\in\Lambda'}$, also covers A, then the sub family $\{A_\lambda\}_{\lambda\in\Lambda'}$ is called* **subcover** *of the open cover $\{A_\lambda\}_{\lambda\in\Lambda}$.* ***Observation:***

1. *Let $A_n = (-n, n)$, where $n \in \mathbb{N}$. Every member of the family $\{A_n\}_{n\in\mathbb{N}}$ is an open interval interval and therefore an open set. The family $\{A_n\}_{n\in\mathbb{N}}$ is an open cover of $\mathbb{R}$. Also, the cover is infinite.*

2. *Let $A'_n = (-2n, 2n)$, where $n \in \mathbb{N}$. Every member of the family $\{A'_n\}_{n\in\mathbb{N}}$ is an open interval and therefore an open set. The family $\{A'_n\}_{n\in\mathbb{N}}$ is an open cover of $\mathbb{R}$. Also, $\{A'_n\}_{n\in\mathbb{N}}$ is a sub-cover of $\{A_n\}_{n\in\mathbb{N}}$.*

(16.3.3) Definition *A subset A of $\mathbb{R}$ is said to be* ***compact*** *if it is closed and bounded.*

(16.3.4) Definition *A subset A of $\mathbb{R}$ is said to have the Heine -Borel property if every open cover of a has a finite sub-cover.*

(16.4) Heine-Borel Theorem

(16.4.1) Theorem *If a set A satisfies Heine- Borel property, then any closed subset of A satisfies Heine- Borel property.*

Proof . *Let A satisfies Heine- Borel property, and B be any closed subset of A. We shall show that B also satisfies Heine- Borel property. Suppose $\{B_\lambda\}_{\lambda\in\Delta}$ is an open cover of B.*

Therefore, $B \subset \bigcup_{\lambda\in\Delta} B_\lambda$

$\Rightarrow B^c \cup B \subset B^c \cup \left(\bigcup_{\lambda\in\Delta} B_\lambda\right)$, *where B^c is open since B is closed.*

$\Rightarrow \mathbb{R} \subset B^c \cup \left(\bigcup_{\lambda\in\Delta} B_\lambda\right)$ *as* $B^c \cup B = \mathbb{R}$

Now $A\subset \mathbb{R}$, *implies*

$A \subset B^c \cup \left(\bigcup_{\lambda\in\Delta} B_\lambda\right)$

$\Rightarrow$ *the family F consisting of* B^c *and* $\{B_\lambda\}_{\lambda\in\Delta}$ *is an open cover of A.*

But A has Heine- Borel property, implies F has a finite subcover say, G consisting of B^c *and* $B_{\lambda_1}, B_{\lambda_2}, B_{\lambda_3}, ..., B_{\lambda_n}$. *Since* $B \subset A$, *it implies that*

$B \subset B^c \cup B_{\lambda_1} \cup B_{\lambda_2} \cup B_{\lambda_3} \cup ..... \cup B_{\lambda_n}$

Again, $B \cap B^c = \phi$, *it follows that*

$B \subset B_{\lambda_1} \cup B_{\lambda_2} \cup B_{\lambda_3} \cup ..... \cup B_{\lambda_n}$

$\Rightarrow \{B_{\lambda_1}, B_{\lambda_2}, B_{\lambda_3}, ..., B_{\lambda_n}\}$ *is an open cover of B.*

Thus, the open cover $\{B_\lambda\}_{\lambda\in\Delta}$ *of B has a finite sub cover* $\{B_{\lambda_1}, B_{\lambda_2}, B_{\lambda_3}, ..., B_{\lambda_n}\}$. *Hence, B also satisfies Heine- Borel property. This completes the proof.*

(16.4.2) Theorem ***HEINE-BOREL THEOREM*** *A set A is compact if and only if A has the Heine- Borel property.*

Proof . *Assume that A be a compact set. Then A is bounded and closed. Let* a = *g.l.b A and b= l.u.b A. Therefore,* $A \subset [a, b]$.

If a=b, then $A = \{a\}$ *and ever open cover of A contains nbd. of a. This nbd. is then the finite sub cover. Thus, A has the Heine- Borel Property.*

Now, let $a \neq b$ *and* $[a, b] = I$. *We shall prove that I satisfies the Heine Borel Property. Suppose I does not have the Heine Borel Property. Then there exists a family F of open sets which covers I, but no finite sub family of which covers I. Divide I into two equal closed intervals* I' *and* I'', *where* $I' = \left[a, \frac{a+b}{2}\right]$ *and* $I'' = \left[\frac{a+b}{2}\right]$. *Then at least one of these* I' *and* I'' *cannot be covered by finitely many members of F. Let* I_1 *be that one of* I' *and* I'' *which is not covered by finitely many members of F. Length of* $I_1 = l\left(I_1\right) = \frac{1}{2}\left(b - a\right)$. *Again divide* I_1

into two equal closed intervals I_1'and I_1". Then atleast one of these cannot be covered by finitely many members of F. Let I_2 be that one of I_1'and I_1" which is not covered by fnitely many members of F. Length of $I_2 = l\left(I_2\right) = \frac{1}{2^2}\left(b - a\right)$. Continuing this way, we get a sequence $\{I_n\}$ of closed intervals such that

i No I_n can be covered by finitely many members of F.

ii $I = I_0 \supset I_1 \supset I_1 \supset I_2 \supset ...$
That is, $I_{n+1} \subset I_{n,\forall n}$.

iii Length of $I_n = l\left(I_n\right) = \frac{1}{2^n}\left(b - a\right) \to \ 0$ as $n \to \infty$.

Therefore, by Nested Interval Property of Sequences, $\bigcap_n^I$ is a singleton. Let $\bigcap_n^I = \{x\}$, then $x \in I$. Since the family F is an open cover of I, there is an open set B $\in$F such that x$\in$ B
$\Longrightarrow$ *B is nbd. of x*
$\Longrightarrow$ *$\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset B$.*
Now, Length of $I_n = l\left(I_n\right) \to \ 0$ as $n \to \infty$. Therefore, there is a natural number m such that $l\left(I_m\right) \ < \epsilon$ and $I_m \subset (x - \epsilon, x + \epsilon) \subset B$ Thus, I_m is covered by a single member B of F, which is a contradiction (since no I_n can be covered by finitely many members of F). Therefore, our supposition was wrong. This implies, I has Hiene-Borel Property. Since A is a closed subset of I, therefore A also has the Hiene-Borel property.
Conversely,let A have the Hiene Borel Property. We shall show that A is closed and bounded. We know that$\{I_n\}$, where $I_n = (-n, n)$ of open intervals cover $\mathbb{R}$. But A has Heine-Borel property implies, there exist finitely many natural numbers $n_1, n_2, n_3, ..., n_k$ such that the finite family $\{I_{n_1}, I_{n_2}, ..., I_{n_k}\}$ covers A. If M = max. $n_1, n_2, n_3, ..., n_k$, then A$\subset$ $(-M, M)$. Hence, A is bounded.

Suppose A is not closed. Then there exists an infinite subset B of A which has no limit point in A. Let $x \in A$ and $x \notin B$, that is, $x \in A \setminus B$. Since x is not a limit point of B, there exists an open interval G_x around x which doesnot contain any point of B. Let $y \in B$, then y is not a limit point of B. There exists an open interval H_y around y containing only one point, namely y, of B. Clearly, H_y is infinite in number as B is infinite. Since the points belonging to A are either in $A \setminus B$ or in B. Therefore, the family of open intervals G_x and H_y forms an open cover of A. This family has no finite sub-cover, because if we omit say, H_y, the corresponding point y is left uncovered. This is a contradiction, because A has the Heine- Borel property, therefore every open cover of A must have a finite sub-cover.

Therefore, our supposition is wrong. Hence, A is closed.

(16.5) Let Us Sum Up: *On real line any closed and bounded set is compact. However, this is not true for every metric space.*

(16.6) Lesson End Exercise

1. *Which of the following sets are compact*

   i. $[0, 1] \cup [3, 4]$

   Sol. *The set* $[0, 1] \cup [3, 4]$ *being the finite union of closed and bounded intervals is a closed and bounded set and therefore it is compact.*

   ii. $\mathbb{N}$

   Sol. *The set of natural numbers is not compact as it is closed but not bounded.*

iii. $A = \left\{1^3, 2^3, 3^3, ..., (132)^3\right\}$

Sol. *Since A is a finite subset of $\mathbb{R}$, A is closed and bounded. Thus, A is compact.*

2. Show that finite union of compact sets is compact.

Sol. *Let $\{A_1, A_2, ..., A_n\}$ be a finite family of compact sets. Then each A_i is a closed and bounded set, $1 \leq i \leq n$.*

Let $S = \bigcup_{i=1}^{n} A_i$.

Since the union of finite family of closed sets is closed, it follows that S is a closed set.

Also, $A_i \subset [a_i, b_i]\,, \quad 1 \leq i \leq n$.

If $a = min.\{a_1, a_2, a_3, ..., a_n\}$

and $b = max.\{b_1, b_2, b_3, ..., b_n\}$

then $S \subset [a, b]$

$\Rightarrow$ S is bounded.

Now S is closed and bounded, implies S is compact.

Hence the proof.

3. Show that arbitrary intersection of compact sets , containing atleast one point in common is compact.

Sol. *Let $\{A_\lambda\}\,da \in_{\lambda \in \delta}$ be an arbitraray family of compact sets. Then each A_λ is closed and bounded for $\lambda \in \delta$.*

Let $S = \bigcap_{\lambda \in \delta} A_\lambda$.

Since the intersection of an arbitrary family of closed sets is a closed set. Therefore, S is a closed set.

Also $S \subset A_\lambda, \quad \forall \lambda \in \delta$ and each A_λ is bounded.

Therefore, S is bounded.

Now S is closed and bounded, implies S is compact.

Hence the proof.

4. *Given the set S=* $\{1, 1.1, 0.9, 1.01, 0.99, 1.001, 0.999, ...\}$

   a *Is the set S bounded?*

   b *Does the set S have l.u.b and g.l.b? If so, determine them.*

   c *Does the set S attains its bounds?*

   d *Find the interior of S?*

   e *Does the set S have any limit point? If so, determine them.*

   f *Is S closed?*

   g *Is S a compact set?*

Sol.

a *S=* $\{1\} \cup \left\{1 \pm \frac{1}{10^n} : n \in \mathbb{N}\right\} \subset [0.9, 1.1]$
$\Rightarrow$ *S is bounded.*

b *S is non- empty bounded subset of* $\mathbb{R}$

Therefore, S has the l.u.b and the g.l.b.

1.1 is an upper bound of S and 1.1 $\in$ *S.*

$\Rightarrow$ *l.u.b S =1.1*

0.9 is a lower bound of S and 0.9 $\in$ *S*

$\Rightarrow$ *g.l.b S =0.9*

c *Yes*

d *Let* $x \in S$. *For any* $\epsilon > 0$, $(x-\epsilon, x+\epsilon)$ *is a nbd. of* x. *Since* $(x-\epsilon, x+\epsilon)$ *contains infinitely many points which are not in* S. *Therefore,* S *is not a nbd. of* x.

$\Rightarrow S$ *is not a nbd. of any of its points.*

$\Rightarrow S^o = \phi$.

e *Yes,* S *has one limit point, namely 1.*

f *Since the only limit point* $1 \in S$, *implies* S *is closed.*

g S *is closed and bounded implies* S *is compact.*

5 *Which of the following are compact?*

a $A = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a \neq b\right\}$

b $B = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \neq 1, a \neq b\right\}$

c $C = \{(x,y) \in \mathbb{R}^2 : ax + by + 5 = 0\}$

d $D = \{(x,y) \in \mathbb{R}^2 : ax = by^2\}$

e $E = \{(x,y) \in \mathbb{R}^2 : x^3 + y^3 = 1\}$

Sol:

a A *is the boundary of an ellipse hence it is closed and bounded*

$\Rightarrow A$ *is compact.*

b B *is an interior of an ellipse hence it is closed and bounded*

$\Rightarrow B$ *is compact.*

c *C is a plane in* $\mathbb{R}^3$*, hence unbounded*

$\Rightarrow$ *C is not compact.*

d *D is a parabola and hence not bounded*

$\Rightarrow$ *D is not compact.*

e *E is not bounded*

$\Rightarrow$ *E is not compact.*

6 Which of the following subsets of $\mathbb{R}^2$ *are compact?*

1. $\{(x, y) : |x| \leq 1, |y| \leq 1\}$
2. $\{(x, y) : |x| \leq 1, |y^2| \leq 1\}$
3. $\{(x, y) : x^2 + 3y^2 \leq 5\}$
4. $\{(x, y) : x^2 \leq y^2 + 1\}$

Sol.

1,2. The sets 1 and 2 are the interior of the square with boundary formed by lines $x = \pm 1$ *and* $y = \pm 1$*, hence are closed and bounded.*

$\Rightarrow$ *1 and 2 are compact.*

3. The set 3 is the interior of an ellipse with boundary hence closed and bounded

$\Rightarrow$ *3 is compact.*

4. The set 4 is the boundary of a hyperbola which is unbounded

$\Rightarrow$ *4 is not compact.*

7. Which of the following sets are compact?

a. $\{(x, y) \in \mathbb{R}^2 \, |x^2 + y^2| \leq 1\}$

b. $\{(x, y) \in \mathbb{R}^2 \, |x^2 + y^2| \geq 1\}$

c. $\{(x, y) \in \mathbb{R}^2 \, |x^2 + y^2| < 1\}$

d. $\{(x, y) \in \mathbb{R}^2 \, |x^2 + y^2| = 1\}$

Sol

a. *The set a. is the interior and boundary of the unit circle in* $\mathbb{R}^2$*, hence it is closed and bounded*

$\Rightarrow$ *a. is compact.*

b. *The set b. is the exterior of the unit circle in* $\mathbb{R}^2$*, hence it is unbounded*

$\Rightarrow$ *b. is not compact.*

c. *The set c. is the interior of the unit circle in* $\mathbb{R}^2$*, hence it is not closed*

$\Rightarrow$ *c. is not compact.*

d. *The set d. is the unit circle in* $\mathbb{R}^2$*, hence it is closed and bounded*

$\Rightarrow$ *d. is compact.*

(16.7) University Model Questions

1. *State and prove Heine-Borel Theorem.*

2. *Show that a closed subset of a compact set is compact.*

(16.8) Suggested Readings

1 *T. M. Apostol, Calculus (Vol. I), John Wiley and Sons (Asia) P. Ltd., 2002*

2. *S. C. Malik and S. Arora, Mathematical Analysis, New Age international Publishers, 2010.*

3. *R.G. Bartle and D. R Sherbert, Introduction to Real Analysis, John Wiley and Sons (Asia) P. Ltd.,2000.*

Lesson-XVII Bolzano-Weirstrauss Theorem

(17.1) Introduction *The Bolzano- Weierstrass Theorem says something intutive: that a set of numbers of infinite cardinality yet whose elements are bounded in size, is going to have a huddle around at least one point.*

(17.2) Objectives *In lesson XV, we have seen that a finite set has no limit point. Also, we have seen that an infinite set may or may not have a limit point. In this lesson, we shall study Bolzano- Weierstrass Theorem, which sets out sufficient conditions for a set to have a limit point. The main aim of this lesson is to introduce Bolzano- Weierstrass Theorem to the students.*

(17.3) Bolzano-Weierstrass Theorem

(17.3.1) Theorem Bolzano-Weierstrass Theorem *Every infinite bounded set has a limit point.*

Proof *: Let S be any infinite bounded set and m, M its infimum and supremum respectively. Let P be a set of real numbers defined as follows:*

$$\{x : x \; exceeds \; at \; the \; most \; a \; finite \; number \; of \; members \; of \; S\}.$$

Clearly, P is non empty as $m \in P$. Also, M is an upper bound of P, for no number greater than or equal to M can belong to P. Thus the set P is non-

empty and is bounded above. Therefore, by the order completeness property, P has the supremum say l. We shall show that l is the limit point of S. Consider any nbd. $(l-\epsilon, l+\epsilon)$ *of l, where* $\epsilon > 0$.

Since l is the supremum of P, there exists at least one member say q of P such that $q > l - \epsilon$. *Since* $q \in P$, *therefore it exceeds at most a finite number of members of S and so* $l-\epsilon$ *can exceed at most a finite number of members of S. Also,* $l+\epsilon$ *exceeds infinitely many members of S, implies* $(l-\epsilon, l+\epsilon)$, *contains infinite members of S. This proves that l is a limit point of S. Hence the proof.*

Note: *Boundedness is not necessary in order for an infinite set S to have a limit point. The unbounded interval* (a, ∞) *has infinitely many limit points.*

(17.3.2) Theorem *Prove that a finite set has no limit point.*

Proof. *Let* $A=\{x_1, x_2, ..., x_n\}$ *be a finite subset of* $\mathbb{R}$. *If possible, assume that A has a limit point say x.*

Now, if we choose $\epsilon = min. \{|x-x_1|, |x-x_2|, ..., |x-x_n|\}$, *then* $(x-\epsilon, x+\epsilon)$ *is a nbd. of x which contains no point of A, a contradiction. Hence our supposition was wrong. Since x is arbitrary, it follows that A has no limit point.*

(17.3.3) Theorem *Prove that 0 is the limit point of set*

$$S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}.$$

Proof . *For each* $\epsilon > 0$, $(-\epsilon, \epsilon)$ *is a nbd. of 0. By Archimedean property of reals, for each* $\epsilon > 0, \exists n \in \mathbb{N}$ *such that* $n > \frac{1}{\epsilon}$

$\Rightarrow \quad \frac{1}{n} < \epsilon$

$\Rightarrow \quad -\epsilon < 0 < \frac{1}{n} < \epsilon$

$\Rightarrow \quad \frac{1}{n} \in (-\epsilon, \epsilon)$

Thus every nbd. of 0 contains a point of S, namely $\frac{1}{n}$.

$\Rightarrow$ *0 is the limit point of S.*

Uniqueness.

$S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subset (0, 1]$.

We shall show that there is no real number other than 0 which is a limit point of S. Let x be a non- zero real number. Then the following cases arise:

Case(i)

If $x < 0$, *then* $(-\infty, 0)$ *is a nbd. of x which contains no point of S.*

$$i.e., (-\infty, 0) \cap S = \phi.$$

Therefore, x is not a limit point of S.

Case(ii)

If $x > 1$, *then* $(1, \infty)$ *is a nbd. of x which does not contain any point of S.*

$$i.e., (1, \infty) \cap S = \phi.$$

Therefore, x is not a limit point of S.

Case(iii)

If $x = 1$, *then* $\left(\frac{1}{2}, \infty\right)$ *is a nbd. of x which does not contain any point of S.*

$$\left(\frac{1}{2}, \infty\right) \cap S - \{1\} = \phi.$$

Therefore, x is not a limit point of S.

Case(iv)

If $0 < x < 1$, *then* $\frac{1}{x} > 0$

Therefore, there exists a unique natural number n such that

$n \leq \frac{1}{x} < n + 1$

$\Rightarrow \quad \frac{1}{n} \geq x > \frac{1}{n+1}$

$\Rightarrow \quad \frac{1}{n+1} < x \leq \frac{1}{n}\frac{1}{n-1}$

$\Rightarrow$ *the nbd.* $\left(\frac{1}{n+1}, \frac{1}{n-1}\right)$ *of x contains only one point* $\frac{1}{n}$*of S, i.e., only finite* *number of points of S.*

Hence 0 is the only limit point of S.

(17.3.4) Theorem *Prove that for any set A, $A^{'}$ is a closed set.*

Proof *To prove that $A^{'}$ is a closed set, we shall show that $\left(A^{'}\right)^{c}$ is an open set. for this, let $x \in \left(A^{'}\right)^{c}$.*

$\Rightarrow \quad x \notin A^{'}$

$\Rightarrow$ *x is not a limit point of A.*

$\Rightarrow \exists$ *a nbd. $I = (x - \epsilon, x + \epsilon)$ of x such that*

$I \cap A - \{x\} = \phi$.

Let $y \in I$, then I being an open interval is an open set.

$\Rightarrow$ *I is a nbd. of y. Also, $I \cap A - \{x\} = \phi$.*

$\Rightarrow$ *y is not a limit point of A.*

$\Rightarrow y \notin A^{'} \Rightarrow y \in \left(A^{'}\right)^{c}$.

Now, $y \in I \Rightarrow y \in \left(A^{'}\right)^{c}$.

Therefore, $I = (x - \epsilon, x + \epsilon) \subset \left(A^{'}\right)^{c}$.

$\Rightarrow \left(A^{'}\right)^{c}$ *is a nbd. of x.*

Since x is any element of $\left(A^{'}\right)^{c}$, it follows that $\left(A^{'}\right)^{c}$ is nbd. of each of its points. This proves that $\left(A^{'}\right)^{c}$ is open. Hence, $A^{'}$ is a closed set.

(17.3.5) Theorem *If $A, B \subset \mathbb{R}$, then*

i $A \subset B \Rightarrow A^{'} \subset B^{'}$

ii $(A \cup B)^{'} = A^{'} \cup B^{'}$

iii $(A \cap B)^{'} \subset A^{'} \cap B^{'}$

Proof.

i. *If* $A^{'} = \phi$, *then* $A^{'} \subset B^{'}$, *since empty set is the subset of every set.*

If $A^{'} \neq \phi$, *let* $x \in A^{'}$ *and* N *be any nbd. of* x.

$\Rightarrow$ *N contains infinitely many points of A.*

$\Rightarrow$ *N contains infinitely many points of B.*

$\Rightarrow$ *x is a limit point of B. That is,* $x \in B^{'}$.

Now $x \in A^{'}$ *implies* $x \in B^{'}$.

Therefore, $A^{'} \subset B^{'}$.

ii. *Since* $A \subset A \cup B$ *and* $A \subset A \cup B$

$\Rightarrow A^{'} \subset (A \cup B)^{'}$ *and* $B^{'} \subset (A \cup B)^{'}$

$\Rightarrow A^{'} \cup B^{'} \subset (A \cup B)^{'}$ *...1*

Now we proceed to show that $(A \cup B)^{'} \subset A^{'} \cup B^{'}$.

If $(A \cup B)^{'} = \phi$, *then* $(A \cup B)^{'} \subset A^{'} \cup B^{'}$.

If $(A \cup B)^{'} \neq \phi$, *letx* $\in (A \cup B)^{'}$.

$\Rightarrow$ *x is a limit point of* $A \cup B$.

$\Rightarrow$ *every nbd. of x contains infinitely many points* $A \cup B$.

$\Rightarrow$ *every nbd. of x contains infinitely many points of A or B.*

$\Rightarrow$ *x is a limit point of A or a limit point of B.*

$\Rightarrow x \in A^{'}$ *or* $x \in B^{'}$

$\Rightarrow x \in A^{'} \cup B^{'}$

Since $x \in (A \cup B)^{'}$, *implies* $x \in A^{'} \cup B^{'}$ *....2*

From 1 and 2 we have

$$(A \cup B)^{'} = A^{'} \cup B^{'}.$$

iii. $A \cap B \subset A$ *implies* $(A \cap B)^{'} \subset A^{'}$

$A \cap B \subset B$ *implies* $(A \cap B)^{'} \subset B^{'}$

Therefore, $(A \cap B)^{'} \subset A^{'} \cap B^{'}$.

Note: $(A\cap B)^{'}$ *and* $A^{'}\cap B^{'}$ *may not be equal. For example, let* $A=(1,2)$, $B=(2,3)$ *. Therefore,* $A\cap B=\phi$*, implies* $(A\cap B)^{'}=\phi$.
Also, $A^{'}=[1,2]$, $B^{'}=[2,3]$.
$A^{'}\cap B^{'}=\{2\}$.
Therefore, $(A\cap B)^{'}\neq A^{'}\cap B^{'}$.

(17.3.6) Theorem *Prove that the derived set of an infinite bounded subset of* $\mathbb{R}$ *is bounded.*

Proof *Let* S *be an infinite bounded subset of* $\mathbb{R}$, *then there exist real numbers* h,k *such that* $S\subset[h,k]$.
Since S *is infinite and bounded* $S^{'}\neq\phi$. $[By\ Bolzano\ Weierstrass\ Theorem]$
. *We shall show that no element of* $S^{'}$ *is less than* h *r greater than* k.
If $x<h$*, then for* $\epsilon=h-x>0$, $(x-\epsilon,x+\epsilon)$ *is a nbd. of* x *containing no element of* $[h,k]$ *and hence containing no element of* S.
Therefore, $x\notin S^{'}$
If $x>k$*, then for* $\epsilon=x-k>0$, $(x-\epsilon,x+\epsilon)$ *is a nbd. of* x *containing no element of* $[h,k]$ *and hence containing no elements of* S.
Therefore, $x\notin S^{'}$
Thus, $x\notin[h,k]$
$\Rightarrow\quad x\notin S^{'}$
$\Rightarrow$ *all the limit points of* S *lie in* $[h,k]$
$\Rightarrow\quad S^{'}\subset[h,k]$
$\Rightarrow\quad S^{'}$ *is bounded.*

(17.4) Let Us Sum Up *Bolzano-Weierstrass Theorem gives us a sufficient condition for an infinite set to have a limit point. Bolzano-Weierstrass Theorem is one of the most fundamental theorem in real analysis and is closely* *related to Heine- Borel Theorem and Cantor's Intersection Theorem, each of which can be easily derived from either of other two.*

(17.5) Lesson End Exercise

i. Give one example of each of the following:

1. *An infinite set having no limit point.*
2. *An infinite set having one limit point.*
3. *A set having two limit points.*
4. *A set having infinite number of limit points.*
5. *A set every point of which is a limit point.*
6. *A set with only* $\sqrt{3}$ *as a limit point.*
7. *A set with only 0 as limit point.*
8. *A unbounded subset of* $\mathbb{R}$ *with limit points.*

ii. Find the derived set of each of the following:

a. $(1, \infty)$

b. $(-\infty, -1)$

c. $\left\{\frac{1+(-1)^n}{n}\right\}$

d. $\left\{r\sqrt{2} : r \in \mathbb{Q}\right\}$

Sol.i.

1. $\mathbb{N}, \mathbb{Z}$ *are infinite sets having no limit point.*

2. The set $\left\{\frac{1}{n}, n \in \mathbb{N}\right\}$ *has only one limit point* 0.

3. The set $\left\{\frac{1}{n}, n \in \mathbb{N}\right\} \cup \left\{1 + \frac{1}{n}, n \in \mathbb{N}\right\}$ *has two limit points* 0 *and* 1.

4. The sets $\mathbb{Q}, \mathbb{R}, [1,2], (2,3)$ *have infinite number of limit points.*

5. Every point of $\mathbb{R}, [1,2]$ *is a limit point.*

6. The set $\left\{\sqrt{3} + \frac{1}{n}, n \in \mathbb{N}\right\}$ *has only* $\sqrt{3}$ *as a limit point.*

7. The set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ *has only 0 as a limit point.*

8. The set $\mathbb{Q}$ *of rational numbers is a subset of* $\mathbb{R}$ *which is unbounded but each point of* $\mathbb{R}$ *is a limit point of* $\mathbb{Q}$.

For any r$\in \mathbb{Q}$, *we have* $\epsilon > 0$ *such that*

$$(r - \epsilon, r + \epsilon) \cap \mathbb{Q} \neq \phi$$

Sol. ii.

a. Let x *be any real number.*

If x¡1, then for $0 < \epsilon < 1 - x$,

$(x - \epsilon, x + \epsilon) \cap (1, \infty) = \phi$.

$\Rightarrow$ *any real number ¡ 1 is not a limit point of* $(1, \infty)$.

If $x \in [1, \infty)$, *then for every* $\epsilon > 0$, $\quad (x - \epsilon, x + \epsilon)$, *contains infinitely many points of* $(1, \infty)$ *to the right of 1.*

$\Rightarrow$ *every element of* $[1, \infty)$ *is a limit point* $(1, \infty)$.

b. Do yourself. c. Let S=$\left\{\frac{1+(-1)^n}{n}\right\}$

When n is odd,

$\frac{1+(-1)^n}{n} = 0.$

When n is even,

$\frac{1+(-1)^n}{n} = \frac{2}{n}$

Therefore, $S = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subset [0, 1]$

Clearly, $S' = \{0\}$.

d. Let $S = \left\{r\sqrt{2} : r \in \mathbb{Q}\right\}$ *Let x be any real number, then for each* $\epsilon > 0$, $(x-\epsilon, x+\epsilon)$ *is a nbd.of x.*

Now $x - \epsilon < x + \epsilon$

$\Rightarrow \quad \frac{x-\epsilon}{\sqrt{2}} < \frac{x+\epsilon}{\sqrt{2}}$

Since between any two distinct real numbers, there are infinitely many rational numbers, therefore there exists infinitely many rational numbers r such that

$\frac{x-\epsilon}{\sqrt{2}} < r < \frac{x+\epsilon}{\sqrt{2}}$

$$\Rightarrow \quad x - \epsilon < r\sqrt{2} < x + \epsilon$$

$\Rightarrow \quad (x-\epsilon, x+\epsilon) \cap S$, *contains infinitely many points of S.*

$\Rightarrow \quad$ *x is a limit point of S.*

Since x is arbitrary, therefore $S' = \mathbb{R}$

(17.6) University Model Questions

1. *State and prove Bolzano- Weierstrass Theorem.*
2. *Show that every infinite bounded set in* $\mathbb{R}$ *has a limit point.*
3. *In each situation below, give an example of a set which satisfies the given condition.*

   a. *A bounded set with no limit point.*

b. An unbounded set with no limit point.

c. An unbounded set with exactly five limit points.

d. A set whose derived set is whole of real line.

4. *Define derived set. Show that the derived set of a set is a closed set.*

5. *Show that if x has a nbd. which contains only finitely many members of a set S, then x cannot be a limit point of S.*

6. *Is it true that if A and B are subsets of $\mathbb{R}$ then $(A \cap B)^{'} = A^{'} \cap B^{'}$? Justify.*

7. *Prove that a finite set has no limit points.*

(17.7) Suggested Readings

1 *T. M. Apostol, Calculus (Vol. I), John Wiley and Sons (Asia) P. Ltd., 2002*

2. *S. C. Malik and S. Arora, Mathematical Analysis, New Age international Publishers, 2010.*

3. *R.G. Bartle and D. R Sherbert, Introduction to Real Analysis, John Wiley and Sons (Asia) P. Ltd., 2000.*

Unit-V

Lesson-XVIII Metric Spaces

(18.1) Introduction: *In the theory of real variables, we had learnt limit, the notion of distance which played important role in defining continuity, convergence and differentiability. In this lesson, we will introduce the generalised notion of distance on arbitrary set called* **metric space** *and illustrate it with examples.*

(18.2) Objectives: *The students will understand how can one define distance on any arbitrary set and generalization of notion of distance between two points of a set.*

(18.3) Metric Spaces

(18.3.1) Definition: *Let X be any set. Then a function*
$d : X \times X \to \mathbb{R}$ *is said to be a metric if*

(*i*) $d(x, y) \geq 0, \forall\, x, y \in X$ *[Non-negative Property]*

(*ii*) $d(x, y) = 0 \Leftrightarrow x = y$

(*iii*) $d(x, y) = d(y, x), \forall\, x, y \in X$ *[Symmetry]* *(iv)* $d(x, y) \leq d(x, z) + d(z, y), \forall\ x, y, z \in X$ *[Trianle Inequality].*

The set X with a metric d is called a metric space. It is denoted by (X, d).

(18.3.2) Definition*: Let X be any set. Then a function*

$d : X \times X \to \mathbb{R}$ is said to be a pseudometric if

(i) $d(x, y) \geq 0, \forall\ x, y \in X$ *[Non-negative Property]*

(ii) $d(x, y) = d(y, x), \forall\ x, y \in X$ *[Symmetry]*

(iii) $d(x, y) \leq d(x, z) + d(z, y), \forall\ x, y, z \in X$ *[Trianle Inequality].*

Note*: Every metric space is pseudo-metric but the converse need not be true.*

Some Results*:*

(i) The absolute value function satisfies the following properties

$$|x| \geq 0,\ |x| = 0 \Leftrightarrow x = 0,\ |x| = |-x|,\ |x + y| \leq |x| + |y|, \forall\, x, y \in \mathbb{R}.$$

(ii) If u and v are complex numbers, then

$$|u + v| \leq |u| + |v|; \frac{|u + v|}{1 + |u + v|} \leq \frac{|u|}{1 + |u|} + \frac{|v|}{1 + |v|}.$$

(iii) **Cauchy-Schwartz Inequality***: Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be any two n-tupple of complex numbers. Then*

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |y_i|^2\right)^{1/2}.$$

(iv) **Minkowski's inequality***: Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be any two n-tupple of complex numbers. Then*

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}, \ if p \geq 1$$

(18.3) Examples

1. *Let $X = \mathbb{R}$, the set of real numbers. Show that the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(x, y) = |x - y|, \forall\, x, y \in \mathbb{R}$ is a metric on $\mathbb{R}$.*

Solution. *We have (i)* $|x-y| \geq 0, \forall\, x,\, y \in \mathbb{R}$

$\Rightarrow d(x,\, y) \geq 0\, \forall,\, x,\, y \in \mathbb{R}$

(ii) $|x-y| = 0 \Leftrightarrow x-y = 0 \Leftrightarrow x = y$ *so that* $d(x,\, y) = 0 \Leftrightarrow x = y$

(iii) $|x-y| = |y-x|\, \forall\, x,\, y \in \mathbb{R} \Rightarrow d(x,\, y) = d(y,\, x)\, \forall\, x,\, y \in \mathbb{R}$

(iv) $|x-y| = |(x-z)+(z-y)| \leq |x-z| + |z-y|\, \forall\, x,\, y,\, z \in \mathbb{R}$

$\Rightarrow d(x,\, y) \leq d(x,\, z) + d(z,\, y)\, \forall\, x,\, y,\, z \in \mathbb{R}.$

Hence from (i)-(iv), it follows that d *is a metric on* $\mathbb{R}$.

Note: *This metric* d *on* $\mathbb{R}$ *is known as usual metric on* $\mathbb{R}$ *and the metric space* $(\mathbb{R},\, d)$ *is known as the usual metric space.*

2. *Let* X *be a non-empty set and define a mapping* $d : X \times X \to \mathbb{R}$ *as*

$$d(x,\, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \quad \forall\, x,\, y \in X$$

Then show that d *is metric on* X.

Solution. *We have (i)* $d(x,\, y) \geq 0$, *by definition of* d.

(ii) $d(x,\, y) = 0 \Leftrightarrow x = y$, *by definition*

(iii) If $x = y$, *then* $d(x,\, y) = 0 = d(y,\, x)$ *and if* $x \neq y$, *then* $d(x,\, y) = 1 = d(y,\, x)$. *Hence* $d(x,\, y) = d(y,\, x)\, \forall\, x,\, y \in \mathbb{R}$

(iv) Let $x,\, y,\, z$ *be any elements in* X. *If* $x = y$, *then* $d(x,\, y) = 0$. *Also* $d(x,\, z) \geq 0$ *and* $d(z,\, y) \geq 0$

Hence $d(x,\, y) \leq d(x,\, z) + d(z,\, y)$

If $x \neq y$, *then either* $x \neq y \neq z$ *or* $x \neq y = z$.

then either $d(x,\, y) = d(x,\, z) = d(z,\, y) = 1$

or $d(x,\, y) = d(x,\, z) = 1$ *and* $d(y,\, z) = 0$

Hence in both situations, $d(x,\, y) \leq d(x,\, z) + d(z,\, y)$

Thus,

$$d(x,\, y) \leq d(x,\, z) + d(z,\, y)\, \forall\, x,\, y,\, z \in X.$$

Hence d is metric on X and $(X,\, d)$ is metric space.

Note: *The metric space $(X,\, d)$ so defined is known as discrete metric space.*

3. *Let $X = \mathbb{R}^2$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x,\, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ where $x = (x_1,\, x_2)$ and $y = (y_1,\, y_2)$. Then show that d is a metric on X.*

Solution. *(i) Since $(x_1 - y_1)^2$ and $(x_2 - y_2)^2$ are non-negative real numbers, we have*

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \geq 0 \Rightarrow d(x,\, y) \geq 0$$

(ii) $d(x,\, y) = 0 \Leftrightarrow \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = 0$

$\Leftrightarrow (x_1 - y_1)^2 = 0$ *and* $(x_2 - y_2)^2 = 0$

$\Leftrightarrow x_1 = y_1$ *and* $x_2 = y_2$

$\Leftrightarrow (x_1,\, x_2) = (y_1,\, y_2) \Leftrightarrow x = y.$

(iii) $d(x,\, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

$= \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$

$= d(y,\, x) \Rightarrow d(x,\, y) = d(y,\, x)\, \forall,\, x,\, y \in X.$

(iv) $d(x,\, y) = \{(x_1 - y_1)^2 + (x_2 - y_2)^2\}^{1/2}$

$= \{\{(x_1 - z_1) + (z_1 - y_1)\}^2 + \{(x_2 - z_2) + (z_2 - y_2)\}^2\}^{1/2}.$

To show that $\{(x_1 - y_1)^2 + (x_2 - y_2)^2\}^{1/2} \leq \{(x_1 - z_1)^2 + (x_2 - z_2)^2\}^{1/2} + \{(z_1 - y_1)^2 + (z_2 - y_2)^2\}^{1/2}.$

Let $\alpha_1 = x_1 - z_1,\, \alpha_2 = x_2 - z_2,\, \beta_1 = z_1 - y_1,\, \beta_2 = z_2 - y_2.$

Then $d(x,\, z) = \sqrt{{\alpha_1}^2 + {\alpha_2}^2}$ *and* $d(z,\, y) = \sqrt{{\beta_1}^2 + {\beta_2}^2}.$

Now, $d(x,\, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

$= \sqrt{((x_1 - z_1) + (z_1 - y_1))^2 + ((x_2 - z_2) + (z_2 - y_2))^2}$

$= \sqrt{(\alpha_1 + \beta_1)^2 + (\alpha_2 + \beta_2)^2}.$

Now $d(x,\, y) \leq d(x,\, z) + d(z,\, y)$

$\Leftrightarrow \sqrt{(\alpha_1 + \beta_1)^2 + (\alpha_2 + \beta_2)^2} \leq \sqrt{{\alpha_1}^2 + {\alpha_2}^2} + \sqrt{{\beta_1}^2 + {\beta_2}^2}$

$\Leftrightarrow (\alpha_1+\beta_1)^2+(\alpha_2+\beta_2)^2$

$\leq {\alpha_1}^2+{\alpha_2}^2+{\beta_1}^2+{\beta_2}^2+2\sqrt{({\alpha_1}^2+{\alpha_2}^2)({\beta_1}^2+{\beta_2}^2)}$

$\Leftrightarrow 2\alpha_1\beta_1+2\alpha_2\beta_2 \leq 2\sqrt{{\alpha_1}^2{\beta_1}^2+{\alpha_1}^2{\beta_2}^2+{\alpha_2}^2{\beta_1}^2+{\alpha_2}^2{\beta_2}^2}$

$\Leftrightarrow 4(\alpha_1\beta_1+\alpha_2\beta_2)^2 \leq 4({\alpha_1}^2{\beta_1}^2+{\alpha_1}^2{\beta_2}^2+{\alpha_2}^2{\beta_1}^2+{\alpha_2}^2{\beta_2}^2)$

$\Leftrightarrow 8\alpha_1\alpha_2\beta_1\beta_2 \leq 4({\alpha_1}^2{\beta_2}^2+{\alpha_2}^2{\beta_1}^2)$

$\Leftrightarrow (\alpha_1\beta_2-\alpha_2\beta_1)^2 \geq 0$ *which is always true.*

Therefore, $\Rightarrow d(x,\, y) \leq d(x,\, z)+d(z,\, y)$.

Hence from (i)-(iv), it follows that d is metric on $\mathbb{R}^2$.

4. *Let* $X=\mathbb{R}^n$ *and* $d : X\times X \to \mathbb{R}$ *be defined by* $d(x,\, y)=\{(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_n-y_n)^2\}^{1/n}$ *where* $x=(x_1,\, x_2,\, \ldots,\, x_n)$ *and* $y=(y_1,\, y_2,\, \ldots,\, y_n)$. *Then show that d is a metric on X.*

Solution. *(i) Since* $(x_1-y_1)^2,\, (x_2-y_2)^2,\, \ldots,\, (x_n-y_n)^2$ *are non-negative real numbers, we have*

$$\{(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_n-y_n)^2\}^{1/2} \geq 0 \Rightarrow d(x,\, y) \geq 0$$

(ii) $d(x,\, y)=0 \Leftrightarrow \{(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_n-y_n)^2\}^{1/2}=0$

$\Leftrightarrow (x_1-y_1)^2=0,\, (x_2-y_2)^2=0,\, \ldots,\, (x_n-y_n)^2=0$

$\Leftrightarrow x_1=y_1,\, \ldots,\, x_n=y_n$

$\Leftrightarrow (x_1,\, x_2,\, \ldots,\, x_n)=(y_1,\, y_2,\, \ldots,\, y_n) \Leftrightarrow x=y$.

(iii) $d(x,\, y)=\{(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_n-y_n)^2\}^{1/2}$

$=\{(y_1-x_1)^2+(y_2-x_2)^2+\ldots+(y_n-x_n)^2\}^{1/2}$

$=d(y,\, x) \Rightarrow d(x,\, y)=d(y,\, x)\, \forall,\, x,\, y \in X$.

(iv) $d(x,\, y)=\{(x_1-y_1)^2+\ldots+(x_n-y_n)^2\}^{1/2}$

$=\{\{(x_1-z_1)+(z_1-y_1)\}^2+\ldots+\{(x_n-z_n)+(z_n-y_n)\}^2\}^{1/2}$

$\leq \{(x_1-z_1)^2+\ldots+(x_n-z_n)^2\}^{1/2}+\{(z_1-y_1)^2+\ldots+(z_n-y_n)^2\}^{1/2}$

by Minkowski's inequality

$\Rightarrow d(x,\, y) \leq d(x,\, z)+d(z,\, y)$.

Hence from (i)-(iv), it follows that d is metric on $\mathbb{R}^n$ and $(\mathbb{R}^n,\, d)$ is a metric space.

5. *If d is a metric space on a non-empty set X then prove that the function*

$$d_1(x,\, y) = \min\{1,\, d(x,\, y)\}\,\forall\, x,\, y \in X$$

is a metric on X.

Solution. *We have (i) $d(x,\, y) \geq 0\,\forall\, x,\, y \in X$*

$\Rightarrow \min\{1,\, d(x,\, y)\} \geq 0$

$\Rightarrow d_1(x,\, y) \geq 0.$

(ii) $d_1(x,\, y) = 0 \Leftrightarrow \min\{1,\, d(x,\, y)\} = 0 \Leftrightarrow d(x,\, y) = 0 \Leftrightarrow x = y.$

(iii) $d_1(x,\, y) = \min\{1,\, d(x,\, y)\} = \min\{1,\, d(y,\, x)\} = d_1(y,\, x).$

(iv) We have to prove that $d_1(x,\, y) \leq d_1(x,\, z) + d_1(z,\, y)$

For this, if $d(x,\, z) = 1$ and $d(z,\, y) = 1$, then the result follows obviously.

Suppose that $d(x,\, z) < 1$ and $d(z,\, y) < 1$.

Then $d_1(x,\, z) + d_1(z,\, y) = d(x,\, z) + d(z,\, y)$

$$\geq d(x,\, y) \geq \min\{1,\, d(x,\, y)\} = d_1(x,\, y)$$

$\Rightarrow d_1(x,\, y) \leq d_1(x,\, z) + d_1(z,\, y).$

Thus from (i)-(iv), it follows that d_1 is metric on X and $(X,\, d_1)$ is a metric space.

6. *Let $(X,\, d)$ be any metric space. Show that the function d_1 defined by*

$$d_1(x,\, y) = \frac{d(x,\, y)}{1 + d(x,\, y)},\ \forall\, x,\, y \in X$$

is a metric on X.

Solution. *Since, $(X,\, d)$ be a metric space.*

Therefore (i) $d(x,\, y) \geq 0 \Rightarrow \frac{d(x,\, y)}{1+d(x,\, y)} \geq 0 \Rightarrow d_1(x,\, y) \geq 0.$

(ii) $d_1(x,\, y) = 0 \Leftrightarrow \frac{d(x,\, y)}{1+d(x,\, y)} = 0 \Leftrightarrow d(x,\, y) = 0 \Leftrightarrow x = y.$

(iii) $d_1(x,\, y) = \frac{d(x,\, y)}{1+d(x,\, y)} = \frac{d(y,\, x)}{1+d(y,\, x)} = d_1(y,\, x).$

(iv) For the triangle inequality, we proceed as follows:

Using the triangle iequality of metric d, we have

$d(x,\, y) \leq d(x,\, z) + d(z,\, y)$

$1 + d(x,\, y) \leq 1 + d(x,\, z) + d(z,\, y)$

$\Rightarrow \frac{1}{1+d(x,\,y)} \geq \frac{1}{1+d(x,\,z)+d(z,\,y)}$

$\Rightarrow 1 - \frac{1}{1+d(x,\,y)} \leq 1 - \frac{1}{1+d(x,\,z)+d(z,\,y)}$

$\Rightarrow \frac{d(x,\,y)}{1+d(x,\,y)} \leq \frac{d(x,\,z)}{1+d(x,\,z)+d(z,\,y)} + \frac{d(z,\,y)}{1+d(x,\,z)+d(z,\,y)}$

$d_1(x,\, y) \leq d_1(x,\, z) + d_1(z,\, y)$.

Therefore from (i)-(iv) it follows d_1 is a metric on X.

7. *Let $X = \mathbb{R}^2$. Then show that a mapping $d : X \times X \to \mathbb{R}$ defined by $d(x,\, y) = |x_1 - y_1| + |x_2 - y_2|$, where $x = (x_1,\, x_2)$, $y = (y_1,\, y_2)$ is a metric on X.*

Solution. *We have (i)* $|x_1 - y_1| \geq 0$, $|x_2 - y_2| \geq 0$

$\Rightarrow |x_1 - y_1| + |x_2 - y_2| \geq 0 \Rightarrow d(x,\, y) \geq 0$.

(ii) $d(x,\, y) = 0 \Rightarrow |x_1 - y_1| + |x_2 - y_2| = 0 \Rightarrow |x_1 - y_1| = 0,\, |x_2 - y_2| = 0$

$\Rightarrow x_1 = y_1,\, x_2 = y_2$

$\Rightarrow x = y$.

(iii) $d(x,\, y) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d(y,\, x)$.

(iv) $d(x,\, y) = |x_1 - y_1| + |x_2 - y_2|$

$$\begin{aligned}
&= |(x_1 - z_1) + (z_1 - y_1)| + |(x_2 - z_2) + (z_2 - y_2)| \\
&\leq |(x_1 - z_1)| + |(z_1 - y_1)| + |(x_2 - z_2)| + |(z_2 - y_2)| \\
&\leq (|(x_1 - z_1)| + |(x_2 - z_2)|) + (|(z_2 - y_2)| + |(z_1 - y_1)|) \\
&\leq d(x,\, z) + d(z,\, y) \Rightarrow d(x,\, y) \leq d(x,\, z) + d(z,\, y).
\end{aligned}$$

Hence from (v)-(iv) it follows that d is a metric on $\mathbb{R}^2$.

(18.4) Let Us Sum Up: *In this lesson we have described the notion of distance on any set called* **metric** *and illustrated with the help of different ex-*

amples. We have observed that the metric on any set is not unique.

(18.5) Lesson end excercise

1. *Let* $X = \mathbb{R}^2$. *Show that the mapping* $d : X \times X \to \mathbb{R}$ *defined by* $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\} \forall x = (x_1, x_2), y = (y_1, y_2)$ *is a metric on* X.

2.*Let* $X = \mathbb{C}$ *be the set of complex number and let* $d : X \times X \to \mathbb{R}$ *be defined by* $d(z_1, z_2) = |z_1 - z_2|, \forall z_1, z_2 \in X$. *Prove that* (X, d) *is a metric space.*

3. *Let* $X = \mathbb{R}$. *Then show that a function* $d : X \times X \to \mathbb{R}$ *defined by* $d(x, y) = \min\{2, |x - y|\}$ *is a metric on* X.

4. *Let* $\mathbb{R}$ *be the set of real numbers. Show that the function* $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *defined by* $d(x, y) = |x^2 - y^2|, \forall x, y \in \mathbb{R}$ *is pseudo-metric on* $\mathbb{R}$ *which is not a metric on* $\mathbb{R}$.

Hint: *Here (i)* $|x^2 - y^2| \geq 0 \Rightarrow d(x, y) \geq 0 \forall x, y \in \mathbb{R}$.

(ii) $d(x, x) = |x^2 - x^2| = 0 \forall x \in \mathbb{R}$

(iii) $d(x, y) = |x^2 - y^2| = |y^2 - x^2| = d(y, x) \forall x, y \in \mathbb{R}$

(iv) $d(x, y) = |x^2 - y^2| = |(x^2 - z^2) + (z^2 - y^2)| \leq |x^2 - z^2| + |z^2 - y^2|$

$\Rightarrow d(x, y) \leq d(x, z) + d(z, y) \forall x, y \in \mathbb{R}$.

This shows that d *is a pseudo-metric on* $\mathbb{R}$.

Now we shall show that d *is not a metric on* $\mathbb{R}$.

For this, we have $d(x, y) = 0 \Rightarrow |x^2 - y^2| = 0 \Rightarrow x^2 - y^2 = 0 \Rightarrow y = +x$ *or* $-x$.

This shows that $d(x, y) = 0$ *does not always imply* $x = y$.

Hence, the function d *is not a metric on* $\mathbb{R}$.

5. *Let* X *be the set of all continuous real- valued functions defined on* $[0, 1]$, *and let*

$$d(x, y) = \int_0^1 |x(t) - y(t)| \, dt, \forall x, y \in X.$$

Show that (X, d) *is a metric space.*

6. *Let* X *be the set of all continuous real- valued functions defined on* $[a, b]$*, and let*

$$d(f, g) = \int_a^b |f(x) - g(x)|\, dt, \forall\, f, g \in X.$$

Show that (X, d) *is a metric space.*

(18.6) University Model Questions

1. *Define metric space. Illustrate this one example.*

2. *Let* $X = \mathbb{R}^2$ *and* $d : X \times X \to \mathbb{R}$ *be defined by* $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ *where* $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$*. Show that* d *is a metric on* X*.*

3. *Let* $X = C[0, 1]$ *be the space of all continuous real valued function on* $[0, 1]$*. Show that the function* $d : X \times X \to \mathbb{R}$ *defined by*

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\}$$

is a metric on X*.*

4. *Let mapping* $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be defined as* $d(x, y) = \frac{|x-y|}{(1+|x-y|)}$*. Prove that* d *is a metric on* $\mathbb{R}$*.*

(18.7) Suggested Readings*: Shanti Narayanan, M. D. Raisinghania; Elements of Real Analysis, S. Chand and Company Pvt. Ltd Ramnagar New Delhi-110055.*

Lesson-XIX Open and Closed Sets in a metric space

(19.1) Introduction: *As we are familiar with open and closed intervals in real line, similarily we can talk of open and closed sets in any set with metric. In this lesson, we shall defined open and closed sets in any metric space.*

(19.2) Objective: *The students will learn the generalisation of open and closed intervals in the real line in the form of open and closed sets in any metric space and their properties.*

(19.3) Open and Closed Sets

(19.3.1) Definition(Open sphere (or Open ball): *Let* (X, d) *be a metric space and* $a \in X$ *be any point. Then the set* $\{x \in X | d(a, x) < r, r > 0\}$ *is* *called an open sphere (or open ball) with centre at* a *and readius* r*. It is denoted by* $S(a, r)$.

(19.3.1) Definition(Closed sphere (or closed ball)*: Let* (X, d) *be a metric space and* $a \in X$ *be any point. Then the set* $\{x \in X | d(a, x) \leq r, r > 0\}$ *is called a closed sphere (or closed ball) with centre at* a *and readius* r*. It is denoted by* $S[a, r]$.

Example. *For the usual metric space* $(\mathbb{R}, d)$*, the open sphere* $S(a, r)$ *is the open interval* $(a - r, a + r)$ *and the closed sphere* $S[a, r]$ *is the closed interval* $[a - r, a + r]$ *where* $a \in \mathbb{R}$ *and* $r > 0$.

(19.3.2) Definition (Neighbourhood of a point). *Let* (X, d) *be a metric space. A set* $N \subset X$ *is said to be a neighbourhood of a point* $a \in X$ *if there exists some* $r > 0$ *such that* $S(a, r) \subset N$.

Example *Let* $(\mathbb{R}, d)$ *be a usual metric space. Then open interval* (a, b) *is a neighbourhood of each of its points.*

(19.4) Open Sets

(19.4.1) Definition (Open Set)*: Let* (X, d) *be a metric space. Then a set* $G \subset X$ *is said to be open if it is a neighbourhood of each of its points.*

Example *Let* $(\mathbb{R}, d)$ *be a usual metric space. Then the open interval* (a, b) *is an open set in* $\mathbb{R}$.

For this, let $x \in (a, b)$ *be any point. Choose* $r < |x - a|$ *and* $r < |b - x|$*. Then* $(x - r, x + r) \subset (a, b)$

$\Rightarrow (a, b)$ *is a neighbourhood of* x*. Since* x *was an arbitrary element of* (a, b)*. Therefore* (a, b) *is a neighbourhood of each of its points. Thus* (a, b) *is an open set.*

(19.4.2) Theorem. *Every open sphere is an open set but the converse need not be true.*

Proof. *Let $S(a, r)$ be an open sphere in a metric space (X, d). Then we have to show that $S(a, r)$ is an open set. For this, let $x \in S(a, r)$. Then $d(a, x) < r$. Now, choose $r' = r - d(a, x)$.*

Claim: $S(x, r') \subset S(a, r)$.

For this, let $y \in S(x, r')$. Then $d(x, y) < r'$.

Now $d(a, y) \leq d(a, x) + d(x, y)$

$< d(a, x) + r'$

$= r$

$\Rightarrow d(a, y) < r$. *Therefore, $y \in S(a, r)$. Hence $S(x, r') \subset S(a, r)$. This shows that each point of $S(a, r)$ is the centre of open sphere contained in it and so $S(a, r)$ is an open set.*

For the converse, let $(\mathbb{R}, d)$ be a usual metric space. Then $(1, 2) \cup (2, 3)$ is an open set but not open sphere.

(19.4.3) Theorem. *Let (X, d) be metric space. A subset of X is open if and only if it is a union of open spheres.*

Proof. *Let A be an open subset of X. Then, for each $x \in A$, there exists a real number $r_x > 0$ such that $x \in S(x, r_x) \subset A$. Then $A \subset \cup\{S(x, r_x) | x \in A\} \subset A$.*

Therefore

$$A = \cup_{x \in A} S(x, r_x).$$

This shows that A is union of open spheres.

Conversely, suppose that $A = \cup_{x \in A} S(x, r_x)$ and let $y \in A$ be any element. Then $y \in S(a, r)$ for some $a \in A$. Since every sphere is open set, so y is the centre of some open sphere $S(y, r')$ such that $S(y, r') \subset S(a, r)$..........(1)

Also, $S(a, r) \subset A$........(2)

Thus from (1) and (2) we have $S(y, r') \subset A$. This shows that A is a neigh-

bourhood of y. Since y was an arbitrary element of A. This implies that A is a neighbourhood of each of its points. Hence A is an open subset of X.

(19.4.4) Theorem. *Let (X, d) be a metric space. Then*

(i) ϕ is open *(ii) X is open*

(iii) the union of arbtrary collection of open sets is open.

(iv) the intersection of finite number of open sets is open.

Proof. *(i) To prove that ϕ is open set we have to prove that ϕ is a nbd of each of its points. Since ϕ has no point so the definition of open set for ϕ is automatically satisfied. Hence ϕ is open set.*

(ii) Let $x \in X$ be any element. Then every open sphere $S(x, r)$ is contained in X. Therefore, X is a nbd of each of its points and X is open set.

(iii) Let $\{G_\alpha | \alpha \in \Delta\}$ be any arbitrary collection of open sets in a metric space X and $G = \cup_{\alpha \in \Delta} G_\alpha$. To show that G is open set, let $x \in G$. Then $x \in G_\lambda$, for some $\lambda \in \Delta$. Since G_λ is open, so there exists $r > 0$ such that $S(x, r) \subset G_\lambda$. This implies that $S(x, r) \subset G_\lambda$ for some $\lambda \in \Delta$.

$$\Rightarrow S(x, r) \subset \cup_{\lambda \in \Delta} G_\lambda.$$

Hence G is open set.

(iv) the intersection of a finite number of open sets is open.

Proof. *Let $\{G_i | i = 1, 2, \ldots, n\}$ be a finite collection of open subsets of X. We wish to show that $\cap_{i=1}^{n} \{G_i\}$ is an open set. For this, if $\cap_{i=1}^{n} \{G_i\} = \phi$, then by (i) $\cap_{i=1}^{n} \{G_i\}$ is open.*

Now, if $\cap_{i=1}^{n} G_i \neq \phi$, let $x \in \cap_{i=1}^{n} G_i$ be any point. Then $x \in G_i, \forall\ i$. Since, each G_i is open, so there exists $r_i > 0$ such that $S(x, r_i) \subset G_i, \forall\, i = 1, 2, \ldots, n$...........(1).

Let $r = \min\{r_i | i = 1, 2, \ldots, n\}$. Then by (1),

we have $S(x, r) \subset S(x, r_i), \forall\ i = 1, 2, \ldots, n$

$\Rightarrow S(x,\, r) \subset G_i,\, \forall\, i = 1,\, 2,\, \ldots,\, n$

$\Rightarrow S(x,\, r) \subset \cap_{i=1}^{n} G_i$. *This shows that* $\cap_{i=1}^{n} G_i$ *is a nbd of each of its points. Hence* $\cap_{i=1}^{n} G_i$ *is an open set.*

(19.5) Closed sets

(19.5.1) Definition (Limit Point): *Let* $(X,\, d)$ *be a metric space and* $S \subset X$. *Then a point* $p \in X$ *is said to be a limit point of* S *if every nbd of* p *contains atleast one point of* S *different from* p. *In other words, a point* $p \in X$ *is said to be a limit point of the subset* $S \subset X$ *if for each* $r > 0$, *the open sphere* $S(p,\, r)$ *contains a point of* S *other than* p *i.e.* $S(p,\, r) \cap S - \{p\} \neq \phi$.

(19.5.2) Definition (Derived Set): *The set of all limit points of a set* $S \subset X$ *is called derived set and is denoted by* S' *or* $D(S)$.

(19.5.3) Definition (Closed Set): *A subset* K *of a metric space* $(X,\, d)$ *is said to be closed if* K *contains all its limit points.*

(19.5.4) Theorem. *Let* $(X,\, d)$ *be a metric space. Then a subset* $K \subset X$ *is closed if and only if* K^c *the complement of* K *is open.*

Proof. *Let us first suppose that* K *is closed set. We shall show that* K^c *is open. For this, if* $K^c = \phi$, *then there is nothing to prove because* ϕ *is open set. Now, we assume that* $K^c \neq \phi$. *Let* $x \in K^c$ *be any element. Then* $x \notin K$ *and* K *is closed* $\Rightarrow$ x *is not a limit point of* K

$\Rightarrow$ *there exists* $r > 0$ *such that* $S(x,\, r) \cap K = \phi$

$\Rightarrow x \in S(x,\, r) \subset K^c$

$\Rightarrow K^c$ *is open.*

Conversely, suppose that K^c *is open set. Then we have to show that* K *is closed. For this, let* $x \in K^c$, *then there exists* $r > 0$ *such that* $S(x,\, r) \subset K^c \Rightarrow S(x,\, r) \cap K = \phi$

$\Rightarrow x$ *is not a limit point of* K

$\Rightarrow K$ *contains all its limit points*

$\Rightarrow K$ *is closed.*

(19.5.5) Theorem. *Every closed sphere is a closed set in a metric space* (X, d).

Proof. *Let* $S[a, r]$ *be a closed sphere with centre at* a *and radius* r. *Let* $G = X - S[a, r]$, *be the complement of* $S[a, r]$. *If* $G = \phi$, *then* G *is open set and hence* $S[a, r]$ *is closed set.*

Now assume that $G \neq \phi$. *Let* $x \in G$. *Then* $d(a, x) > r$, *let* $r' = d(a, x) - r \Rightarrow r' > 0$. *Now, to show that* G *is open, we shall show that* $S(x, r') \subset G$.

For this, let $y \in S(x, r')$ *be any element. Then* $d(x, y) < r'$

$\Rightarrow d(x, y) < d(a, x) - r$

$\Rightarrow r < d(a, x) - d(x, y)$..........$(*)$

Now we know that $d(a, x) \leq d(a, y) + d(y, x)$ *(Trianle inequality)*

$\Rightarrow d(a, x) - d(x, y) \leq d(a, y)$, *Using this in* $(*)$ *we get,*

$r < d(a, y) \Rightarrow y \in G$. *Hence* $S(x, r') \subset G$. *This shows that* G *is open set and* $S[a, r]$ *is a closed set.*

(19.5.6) Theorem. *Let* (X, d) *be a metric space. Then*

(i) ϕ *is closed* $\qquad\qquad$ (ii) X *is closed*

(iii) *The intersection of arbitrary family of closed sets is closed.*

(iv) *The union of finite family of closed sets is closed.*

Proof. (i) *We have* $\phi^c = X$, *which is open set. This implies that* ϕ *is closed set.*

(ii) $X^c = \phi$ *which is open set. This implies that* X *is closed set.*

(iii) *Let* $\{K_\alpha | \alpha \in \Delta\}$ *be an arbtrary collection of closed subsets of* X *and* $K = \cap_{\alpha \in \Delta} K_\alpha$.

Now

$$K^c = (\cap_{\alpha\in\Delta} K_\alpha)^c = \cup_{\alpha\in\Delta} {K_\alpha}^c$$

which is open (because each K_α *is closed set and each* ${K_\alpha}^c$ *is open). Then by Theorem* (19.4.4)(*iii*) *it follows that* K^c *is open set* $\Rightarrow$ K *is closed set. Therefore, any intersection of closed sets in a metric space is closed.*

(*iv*) *Let* $\{K_i | i = 1, 2, 3, \ldots, n\}$ *be finite family of closed subsets of metric space* X *and* $K = \cup_{i=1}^{n} K_i$. *To show that* K *is closed, we have to show that* K^c *is open.*

For this, we have $K^c = \cup_{i=1}^{n} {K_i}^c$

$= \cap_{i=1}^{n} {K_i}^c$ *which is open by Theorem* (19.4.4)(*iv*) *because* ${K_i}^c$ *is open set for each* i. *Hence* K, *the finite union of closed sets in a metric space* X *is closed.*

(19.6) Examples

1. *Find the open sphere* $S(0, \frac{1}{2})$ *in metric space* $(\mathbb{R}, d)$.

Solution. *By definition* $S(0, \frac{1}{2}) = \{x | x \in \mathbb{R} \text{ and } d(x, 0) < \frac{1}{2}\}$

$= \{x | x \in \mathbb{R} \text{ and } |x| < \frac{1}{2}\}$

$= \{x | x \in \mathbb{R} \text{ and } -\frac{1}{2} < \frac{1}{2}\} = (-\frac{1}{2}, \frac{1}{2})$.

2. *Describe open sphere for discrete metric space.*

Solution. *Let* (X, d) *be a metric space. Then*

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \quad \forall\, x, y \in X.$$

Let $a \in X$ *be any element and* $r > 0$ *be a real number. If* $r < 1$, *then* $r = 0$ *(because* $d(x, y)$ *has only two values* 0 *or* 1*)*

and $d(a, x) < r \Rightarrow a = x$. *Therefore,* $S(a, r) = \{a\}$.

If $r > 1$, *then* $S(a, r) = \{x | x \in X \text{ and } d(a, x) < r\} = X$. *Further, if* $r = 1$, *then* $S(a, r) = \{x | x \in X \text{ and } d(a, r) < 1\} = \{a\}$.

3. *Describe open spheres (balls) of radius r and centre a in*

(*i*) *Usual metric space* $(\mathbb{R}, d)$

(*ii*) *Usual metric space* $(\mathbb{R}^2, d)$

(*iii*) *Usual metric space* $(\mathbb{R}^3), d$

(*iv*) *Discrete metric space* $(\mathbb{R}, d)$.

Proof. (*i*) *The required open sphere in the usual metric space* $(\mathbb{R}, d)$ *is given by* $S(a, r) = \{x | x \in \mathbb{R} \text{ and } d(a, x) < r\}$

$$= \{x | x \in \mathbb{R} \text{ and } |x - a| < r\}$$
$$= \{x | x \in \mathbb{R} \text{ and } a - r < x < a + r\}$$
$$= (a - r, a + r).$$

(*ii*) *Here* $d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$,
where $x = (x_1, y_1)$, $y = (x_2, y_2)$. *Therefore*

$$S(a, r) = \{(x_1, y_1) \in \mathbb{R}^2 | \sqrt{(x_1 - a_1)^2 + (y_1 - a_2)^2} < r\}$$
$$= \{(x_1, y_1) \in \mathbb{R}^2 | (x_1 - a_1)^2 + (y_1 - a_2)^2 < r^2\},$$

which is the required open sphere. Hence open sphere $S(a, r)$ *is the interior of circle with centre at* $a = (a_1, a_2)$ *and radius* r.

(*iii*) *Here* $d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$,
where $x = (x_1, y_1, z_1)$, $y = (x_2, y_2, z_2)$. *Therefore* $S(a, r)$

$$= \{(x_1, y_1, z_1) \in \mathbb{R}^3 | \sqrt{(x_1 - a_1)^2 + (y_1 - a_2)^2 + (z_1 - a_3)^2} < r\}$$
$$= \{(x_1, y_1, z_1) \in \mathbb{R}^3 | (x_1 - a_1)^2 + (y_1 - a_2)^2 + (z_1 - a_3)^2 < r^2\},$$

which is the required open sphere. Hence open sphere $S(a, r)$ *is the interior of sphere with centre at* $a = (a_1, a_2, a_3)$ *and radius* r.

(*iv*) *Here*

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \quad \forall\, x, y \in \mathbb{R}.$$

We have two cases:

Case I: *when* $r > 1$, *then the open sphere with centre* $a \in \mathbb{R}$ *and radius* r *is given by*

$S(a,\, r) = \{x | x \in \mathbb{R} \text{ and } d(a,\, x) < r\} = \mathbb{R}$.

Case II: *when* $0 < r \leq 1$,

then $S(a,\, r) = \{x | x \in \mathbb{R} \text{ and } d(a,\, x) < r\} = \{a\}$.

Therefore, the only open spheres in a discrete metric space $(\mathbb{R},\, d)$ *are singelton sets or whole space.*

4. *Let* $(\mathbb{R},\, d)$ *be the usual metric space. Then every open interval is an open set in* $(\mathbb{R},\, d)$ *where as singleton set is not open.*

Solution. *Let* $(a,\, b)$ *be an open interval, where* $a,\, b \in \mathbb{R}$ *and* $a < b$. *To show that* $(a,\, b)$ *is an open set, we have to show that there exists an open shpere with centre at each point of* $(a,\, b)$ *and radius* r *contained in it. For this, let* $x \in (a,\, b)$ *be any point. Choose* $r = \min\{|x - a|,\, |b - x|\}$. *Then there exists an open sphere*

$$S(a,\, r) \subset (a,\, b).$$

Hence every open interval in usual metric space is open set.

Now, let $\{x\}$ *be a singelton set in usual metric space* $(\mathbb{R},\, d)$. *Since every open sphere in usual metric space is an open interval, so any open interval centred at* x *is* $S(x,\, \epsilon) = (x - \epsilon,\, x + \epsilon)$, *where* $\epsilon > 0$ *however small.*

But $S(x,\, \epsilon) = (x - \epsilon,\, x + \epsilon) \not\subset \{x\}$. *Therefore,* $\{x\}$ *is not open set* $\Rightarrow$ *every singelton in usual metric space is not open set.*

5. *Let* $(\mathbb{R},\, d)$ *be the usual metric space. Then every closed interval is a closed set in* $(\mathbb{R},\, d)$.

Solution. *Let* $I = [a,\, b]$ *be a closed interval in the usual metric space* $(\mathbb{R},\, d)$. *Then* $I^c = \mathbb{R} - I = (-\infty,\, a) \cup (b,\, \infty)$, *which is an open set (because the union of open sets is open).*

Hence $[a, b]$ is a closed set.

(19.7) Let Us Sum Up: *In this lesson, we have defined the concepts of open(closed) spheres (or balls), open sets and closed sets. Then the properties of open and closed sets have been discussed. Further, all the concepts have been illustrated with examples.*

(19.8) Lesson end Exercise

1. *Show that every singelton set on the real line with usual metric is closed set.*
2. *Prove that every set in a discrete metric space is an open set.*
3. *Prove that every set in a discrete metric space is an closed set.*

(19.9) University Model Questions

1. *Define open set in a metric space. Prove that every open interval in a usual metric space $(\mathbb{R}, |\ |)$ is an open set.*
2. *Define closed set in a metric space. Prove that every closed interval in a usual metric space $(\mathbb{R}, |\ |)$ is an closed set.*
3. *Prove that every finite set in a metric space (X, d) is closed set.*

Hint: *Let us first show that every singelton set $\{x\}$ is closed in the metric space (X, d). For this, let $G = \{x\}^c$. If $G = \phi$, then G is obviously open set and $\{x\}$ is closed.*

If $G \neq \phi$, then there exists $y \in G \Rightarrow x \neq y$. Let $r = d(x, y)$. Then there exists an open sphere $S(y, r_1)$, where $r_1 < r \Rightarrow x \notin S(y, r_1)$. For this if $x \in S(y, r_1)$, then $d(x, y) < r_1 < r$, a contradiction to the fact that $d(x, y) = r$.

Therefore, $S(y, r_1) \subset \{x\}^c$

$\Rightarrow \{x\}^c$ *is open set. Hence* $\{x\}$ *is closed in* X*. Therefore,* $\{x_1, x_2, \ldots, x_n\} = \cup_{i=1}^{n}\{x_i\}$*, which is a finite union of closed sets. Hence every finite set is closed in* (X, d)*.*

4. *Show that every closed sphere in a metric space is closed set.*

(19.10) Suggested Readings*: Shanti Narayanan, M. D. Raisinghania; Elements of Real Analysis, S. Chand and Company Pvt. Ltd Ramnagar New Delhi-110055.*

Lesson-XX Interior, Closure and Boundary of a Set

(20.1) Introduction: *In a metric space* (X, d)*, it is interesting to study the properties of its subsets i.e, interior , exterior, frontier and boundary points of a set. By knowing these properties of a subset of a metric space, we can derive openness and closedness of a set.*

(20.2) Objectve: *The students will learn to compute explicitly the interior, extrerior, frontier, and boundary points of a set in a metric space.*

(20.3) Interior of a set

(20.3.1) Interior point of a set (Definition): *Let* (X, d) *be a metric space and* $A \subset X$*. Then a point* $a \in A$ *is said to be an interior point of* A *if there exists* $r > 0$ *such that* $S(a, r) \subset A$*.*

The set of all interior points of a set A *is called the* **interior of A** *and is denoted by* A^o*.*

Note: $A^o \subset A$.

(20.3.2) Exterior point of a set (Definition): *Let* (X, d) *be a metric space and* $A \subset X$*. Then a point* $x \in X$ *is said to be an exterior point of* A *if there* *exists $r > 0$ such that $S(a,\, r) \subset A^c$.*

The set of all exterior points of A is called **exterior of the set A** *and is denoted by $ext(A)$ or $(A^c)^o$.*

(20.3.3) Frontier point of a set (Definition): *Let $(X,\, d)$ be a metric space and $A \subset X$. Then a point $x \in X$ is said to be a frontier point of A if it is neither interior point of A nor an exterior point of A. The set of all frontier points of a set A is called* **frontier of A** *and is written as $Fr(A)$.*

Note: $Fr(A) = X - A^o \cup ext(A)$.

(20.3.4) Boundary point of a set (Definition): *Let $(X,\, d)$ be a metric space and $A \subset X$. Then a point $x \in X$ is said to be a boundary point of A if $x \in A$ and x is frontier point of A. The set of all boundary points of A is called* **boundary of** *A. It is denoted by $b(A)$ or $bd(A)$.*

(20.3.5) Adherent point (Definition): *Let $(X,\, d)$ be a metric space and A be a subset of X. Then a point $x \in X$ is said to be an adherent point of A if each open sphere centered at x contains atleast one point of A.*

The set of all adherent points of A is called **closure of A**. *It is denoted by $\overline{A}$.*

(20.3.6) Theorem. *Let (X, d) be a metric space space and $A \subset X$. Then*

(i) A^o is the union of all open subsets of A.

(ii) A is open set if and only if $A^o = A$.

(iii) If $A,\, B \subset X$ such that $A \subset B$, then $A^o \subset B^o$

(iv) A^o is the largest open set contained in A.

Proof. *(i) Let $x \in A^o$. Then there exists $r_x > 0$ such that $S(x,\, r_x) \subset A$. Since each open sphere is open set. Therefore, for each $y \in S(x,\, r_x)$ there exists $r_y > 0$ such that $S(y,\, r_y) \subset S(x,\, r_x) \subset A$ i.e $S(y,\, r_y) \subset A$. Therefore, each point of $S(x,\, r_x)$ is an interior point of A*

$\Rightarrow S(x,\, r_x) \subset A^o\,, \forall\, x \in A^o$

$\Rightarrow \cup_{x\in A^o} S(x,\, r_x) \subset A^o$...........(1)

Also, let $x \in A^o \Rightarrow x \in S(x,\, r_x)$

$\Rightarrow x \in \cup_{x\in A^o} S(x,\, r_x)$

$\Rightarrow A^o \subset \cup_{x\in A^o} S(x,\, r_x)$............(2)

From (1) *and* (2), *we get*

$$A^o = \cup_{x\in A^o} S(x,\, r_x).$$

(ii) First, we suppose that A *is open set. To show that* $A^o = A$*. For this, we have* $A^o \subset A$*.*

Now, since A *is open set, so* $A = \cup_{x\in A} S(x,\, r_x)$*. Let* $y \in A$*, then* $y \in S(x,\, r_x)$*, for some* $x \in A$

$\Rightarrow$ *there exists* r_y *such that* $S(y,\, r_y) \subset S(x,\, r_x) \subset A$

$\Rightarrow S(y,\, r_y) \subset A$*. Therefore,* y *is an interior point of* A *i.e* $y \in A^o$

$\Rightarrow A \subset A^o$*. Hence* $A = A^o$*.*

(iv) Let $x \in A^o$*. Then* x *is an interior point of* A

$\Rightarrow$ *there exists* $r_x > 0$ *such that* $S(x,\, r_x) \subset A$*.*

But $A \subset B$

Therefore $S(x,\, r_x) \subset B$

$\Rightarrow x$ *is an interior point of* B

$\Rightarrow x \in B^o$

$\Rightarrow A^o \subset B^o$*.*

(iv) By (i), we have

$$A^o = \cup_{x\in A^o} S(x,\, r_x)$$

which is any union of open spheres and each open sphere is open set. So, A^o *is open set.*

To show that A^o *is the largest open set, let* B *be any open set contained in* A*. Then* $B^o = B$ *by (ii)*

Now $B \subset A$

$\Rightarrow B^o \subset A^o$

$\Rightarrow B \subset A^o$. *This shows that* A^o *is the largest open set contained in* A.

(20.3.7) Theorem. *Let* (X, d) *be a metric space and let* $A, B \subset X$. *Then* (i) $A^o \cup B^o \subset (A \cup B)^o$(ii) $(A \cap B)^o = A^o \cap B^o$.

Proof. *(i) Since* $A \subset A \cup B$ *and* $B \subset A \cup B$

$\Rightarrow A^o \subset (A \cup B)^o$ *and* $B^o \subset (A \cup B)^o$.

Taking union, we get $A^o \cup B^o \subset (A \cup B)^o$.

(ii) Since $A \cap B \subset A$ *and* $A \cap B \subset B$

$\Rightarrow (A \cap B)^o \subset A^O$ *and* $(A \cap B)^o \subset B^o$

Taking intersection, we get

$(A \cap B)^o \subset A^o \cap B^o$................(1)

Since $A^o \subset A$ *and* $B^o \subset B$.

So, $A^o \cap B^o \subset A \cap B$

$\Rightarrow (A^o \cap B^o)^o \subset (A \cap B)^o$

$\Rightarrow A^o \cap B^o \subset (A \cap B)^o$...........(2).

Therefore, from (1) *and* (2), *we get* $(A \cap B)^o = A^o \cap B^o$.

(20.3.8) Theorem. *Let* (X, d) *be a metric space and* $A \subset X$. *If* x *is an interior point of* A, *then*

$$d(x, A) = glb\{d(x, y) : \forall\, y \in A\} = 0.$$

But the converse is not true.

Proof. *We have* $x \in A^o$ *and* $A^o \subset A$

$\Rightarrow x \in A$.

Now $d(x, A) = glb\{d(x, y) : \forall\, y \in A\}$. *Also* $d(x, y) \geq 0, \forall\, y \in A$ *and* $d(x, x) = 0$ *as* $x \in A$. *Therefore* $0 \in \{d(x, y) : \forall\, y \in A\}$

$\Rightarrow glb\{d(x, y) : \forall\, y \in A\} = 0$.

Hence $d(x,\, A) = 0$.

For the converse, consider $(\mathbb{R},\, d)$ *be the usual metric space and* $A = (0,\, 1)$ *be its subset.*

Then $d(0,\, A) = glb\{d(0,\, y) : \forall\, y \in A\}$

$\Rightarrow d(0,\, A) = glb\{|0 - y| : \forall\, y \in (0,\, 1)\}$

$\Rightarrow d(0,\, A) = glb\{|y| : \forall\, y \in A\}$

Now $\frac{1}{n} \in A, \forall\, n \in \mathbb{N}$ *and* $d(0,\, \frac{1}{n}) = |0 - \frac{1}{n}| = \frac{1}{n} \to 0$ *as* $n \to \infty$

Therefore, $glb\{|y| : \forall\ y \in A\} = 0$ *i.e* $d(0,\, A) = 0$ *but* $0 \notin A$.

(20.3.9) Theorem. *Let* $(X,\, d)$ *be a metric space and* $A,\, B$ *be any subset of* X*, then*

(i) $ext(\phi) = X$ (ii) $ext(X) = \phi$ (iii) $ext(A) \subset A^c$

(iv) *If* $A \subset B$*, then* $ext(B) \subset ext(A)$

(v) $ext(A \cup B) = ext(A) \cup ext(B)$.

Proof. (i) $ext(\phi) = (\phi^c)^O = X^o = X$ *(because the largest open set contained in* X *is* X*).*

(ii) $ext(X) = (X^c)^o = \phi^o = \phi$.

(iii) $ext(A) = (A^c)^o \subset A^c$.

(iv) *Since* $A \subset B \Rightarrow B^c \subset A^c$

$\Rightarrow (B^c)^o \subset (A^c)^o$

$\Rightarrow ext(B) \subset ext(A)$.

(v)
$$\begin{aligned} ext(A \cup B) &= ((A \cup B)^c)^o \\ &= (A^c \cap B^c)^o \\ &= (A^c)^o \cap (B^c)^o \\ &= ext(A) \cap ext(B). \end{aligned}$$

(20.3.10) Theorem. *Let* (X, d) *be a metric space and* $A \subset X$ *then prove that* (i) $A \subseteq \overline{A}$ (ii) $A' \subseteq \overline{A}$ (iii) $\overline{A} = A \cup A'$

(iv) $\overline{A}$ *is closed set.* *(v)* A *is closed if and only if* $\overline{A} = A$.

(vi) $\overline{A}$ *is the smallest closed set containing* A.

(vii) $\overline{A}$ *is the intersection of all closed supersets of* A.

Proof. *(i) Let* $x \in A$.

Then $S(x, r) \cap A \neq \phi$ *for any real number* $r > 0$

$\Rightarrow$ x *is adherent point of* A

$\Rightarrow x \in \overline{A}$.

Therefore $A \subset \overline{A}$.

(ii) Let $x \in A'$. *Then* x *is a limit point of* A

$\Rightarrow S(x, r) \cap A - \{x\} \neq \phi$ *for each* $r > 0$

$\Rightarrow S(x, r) \cap A \neq \phi$ *for each real* $r > 0$

$\Rightarrow x$ *is an adherent point of* A

$\Rightarrow x \in \overline{A}$. *Therefore* $A' \subset \overline{A}$.

(iii) From (i) and (ii) we have $A \subset \overline{A}$ *and* $A \subset A'$

$\Rightarrow A \cup A' \subset \overline{A}$..........(1)

Now, we claim that $\overline{A} \subset A \cup A'$.

For this, let $x \in \overline{A}$. *Suppose that* $x \notin A \cup A'$

$\Rightarrow x \notin A$ *and* $x \notin A'$

$\Rightarrow x \notin A$ *and there exists a real number* $r > 0$ *such that* $S(x, r) \cap A = \phi$ *or* $\{x\}$

$\Rightarrow S(x, r) \cap A = \phi$

$\Rightarrow x$ *is not an adherent point of* A *which contradicts the hypothesis that* $x \in \overline{A}$.

Therefore, our supposition is wrong. Thus $x \in A \cup A'$.

(iv) To prove that $\overline{A}$ *is closed set, we shall prove that* $(\overline{A})^c$ *is open set.*

For this, consider $x \in (\overline{A})^c$. *Then* $x \notin \overline{A}$

$\Rightarrow x$ *is not an adherent point of* A

$\Rightarrow$ *there exists atleast one* $r > 0$ *such that* $S(x, r) \cap A = \phi$.

Now, we claim that $S(x, r) \cap \overline{A} = \phi$.

For this, let $y \in S(x, r)$. *Then* $d(x, y) < r$.

Consider $r' = r - d(x, y)$, *then* $r' > 0$ *and* $S(y, r') \subset S(x, r)$

$\Rightarrow S(y, r') \cap A \subset S(x, r) \cap A = \phi$

$\Rightarrow S(y, r') \cap A \subset \phi \Rightarrow S(y, r') \cap A = \phi$

$\Rightarrow$ *y is not an adherent point of A*

$\Rightarrow y \notin \overline{A}$

$\Rightarrow S(x, r) \cap \overline{A} = \phi$

$\Rightarrow S(x, r) \subset (\overline{A})^c$.

Hence $(\overline{A})^c$ *is open* $\Rightarrow \overline{A}$ *is closed.*

(v) *Suppose that A is closed set. Then we shall show that* $\overline{A} = A$. *For this, we have* $A \subset \overline{A}$ *(by definition).*

Since A is closed set so $A' \subset A$ *and* $A \subset A \Rightarrow A \cup A' \subset A$

$\Rightarrow \overline{A} \subset A$. *Hence* $\overline{A} = A$.

Conversely, suppose that $\overline{A} = A$. *Then by (iv), it follows that A is closed set.*

(vi) *Since* $A \subset \overline{A}$

$\Rightarrow \overline{A}$ *contains A. We have proved in* (iv) *that* $\overline{A}$ *is always closed. Now, let K be any closed set containing A. Then* $A \subset K$

$\Rightarrow \overline{A} \subset \overline{K}$

$\Rightarrow \overline{A} \subset K$.

This shows that $\overline{A}$ *is the smallest closed set containing A.*

(vii) *Let* $F = \cap\{K : K$ *is closed set and* $K \supset A\}$.

Then F is closed.

Since $\overline{A}$ *is closed set containing* $A \Rightarrow \overline{A}$ *is in the above collection and so* $F \subset \overline{A}$.

Since intersection of all closed sets is closed, so F is closed set containing A. But $\overline{A}$ *is the smallest closed set containing* $A \Rightarrow \overline{A} \subset F$. *Thus* $F = \overline{A} \Rightarrow \overline{A}$ *is* *the intersection of closed sets containing A.*

(20.3.11) Theorem. *Let $(X,\, d)$ be a metric space and let $A,\, B$ be any subsets of X. Then (i) if $A \subset B \Rightarrow \overline{A} \subset \overline{B}$ (ii) $\overline{(A \cap B)} \subset \overline{A} \cap \overline{B}$ (iii) $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$.*

Proof. *(i) Let $A \subset B$. Since $B \subset \overline{B}$*

$\Rightarrow A \subset B \subset \overline{B}$

$\Rightarrow \overline{B}$ *is a closed set containing A. But $\overline{A}$ is the smallest closed set containing A. Therefore $\overline{A} \subset \overline{B}$.*

(ii) Since $A \cap B \subset A$ and $A \cap B \subset B$

therefore, by part (i), $\overline{A \cap B} \subset \overline{A}$ and $\overline{A \cap B} \subset \overline{B}$

$\Rightarrow \overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

(iii) Since $A \subset A \cup B$ and $B \subset A \cup B$

therefore, by part (i), we have $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$

$\Rightarrow \overline{A} \cup \overline{B} \subset \overline{A \cup B}$...............(1)

Now, $A \subset \overline{A}$ and $B \subset \overline{B}$

$\Rightarrow A \cup B \subset \overline{A} \cup \overline{B}$

$\Rightarrow \overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}}$

$\Rightarrow \overline{A \cup B} \subset \overline{A} \cup \overline{B}$............(2) *(because $\overline{A} \cup \overline{B}$ is a closed set)*

From (1) and (2), we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(20.4) Examples

1. *Let $(\mathbb{R},\, d)$ be the usual metric space. Find the interior, exterior, frontier and boundary points of each of the folowing subsets of $\mathbb{R}$:*

(a) $(0,\, 1)$ *(b)* $\{\frac{1}{n} : n \in \mathbb{N}\}$ *(iii)* $\mathbb{Q}$.

Proof. *(a) Let $A = (0,\, 1)$*

(i) Clearly $(0,\, 1)$ is open set. Therefore $A^o = A = (0,\, 1)$.

(ii) $Ext(A) = (A^c)^o = ((-\infty,\, 0] \cup [1,\, \infty))^o$

$= (-\infty, 0]^o \cup [1, \infty)^o$

$= (-\infty, 0) \cup (1, \infty).$

(iii) $Fr(A) = \mathbb{R} - A^o \cup Ext(A) = \mathbb{R} - (0, 1) \cup (A^c)^o$

$= \mathbb{R} - (0, 1) \cup (-\infty, 0) \cup (1, \infty)$

$= \mathbb{R} - \mathbb{R} = \phi.$

(iv) *Since* $bd(A) \subset Fr(A)$

$\Rightarrow bd(A) = \phi.$

(b) (i) *Let* $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. *Then* $A^o = \phi$ *because there does not exist open interval containing any point say* $\frac{1}{m} \in A$, *where* $m \in N$ *such that*

$$\left(\frac{1}{m} - \epsilon, \frac{1}{m} + \epsilon\right) \subset A.$$

(ii) $Ext(A) = (A^c)^o = \big(\mathbb{R} - \{\frac{1}{n} : n \in \mathbb{N}\}\big)^o$

$= \mathbb{R} - \{\frac{1}{n} : n \in \mathbb{N}\}.$

(iii) $Fr(A) = \mathbb{R} - A^o \cup Ext(A)$

$= \mathbb{R} - \phi \cap \big[\big(\mathbb{R} - \{\frac{1}{n} : n \in \mathbb{N}\}\big)^o\big]^c$

$= \mathbb{R} \cap \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$

$= \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}.$

(iv) $bd(A) = A - A^o \cup Ext(A)$

$= \{\frac{1}{n} : n \in \mathbb{N}\} - \big[\mathbb{R} - \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}\big]$

$= \{\frac{1}{n} : n \in \mathbb{N}\}$

(c) (i) *Let* $x \in \mathbb{Q}$. *Then there does not exist an open sphere with centre* $x \in \mathbb{Q}$ *and contained in* $\mathbb{Q}$. *Therefore,* $\mathbb{Q}^o = \phi$.

(ii) $Ext(\mathbb{Q}) = (\mathbb{R} - \mathbb{Q})^o = (Ir)^o = \phi.$

(iii) $Fr(\mathbb{Q}) = \mathbb{R} - \mathbb{Q}^o \cup Ext(\mathbb{Q}) = \mathbb{R} - \phi \cup \phi = \mathbb{R}.$

(iv) $bd(\mathbb{Q}) = \mathbb{Q} - \mathbb{Q}^o \cup Ext(\mathbb{Q}) = \mathbb{Q} - \phi \cup \phi = \mathbb{Q}.$

2. *Find the closure of the following subset of* $\mathbb{R}$ *in usual metric space* (i) *singleton set* (ii) *finite subset of* $\mathbb{R}$ (iii) $\mathbb{N}$ (iv) $\mathbb{Z}$ (v) $\mathbb{Q}$ (vi) $\mathbb{R} - \mathbb{Q}$.

Solution. (i) *Let* $A = \{x\}$*. Then the derived set of* A *is given by* $A' = \phi$*. Therefore,* $\overline{A} = A \cup A' = A \cup \phi = A$*. i.e,* $\overline{\{x\}} = \{x\}$*.*

(ii) *Let* $A = \{x_1, x_2, \ldots, x_n\}$ *be a finite set. Then* $A' = \phi$*. Now,* $\overline{A} = A \cup A' = A \cup \phi = A$*.*

(iii) $\overline{\mathbb{N}} = \mathbb{N} \cup \mathbb{N}' = \mathbb{N} \cup \phi = \mathbb{N}$.

(iv) $\overline{\mathbb{Z}} = \mathbb{Z} \cup \mathbb{Z}' = \mathbb{Z} \cup \phi = \mathbb{Z}$.

(iv) $\overline{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}' = \mathbb{Q} \cup \mathbb{R} = \mathbb{R}$.

(iv) $\overline{\mathbb{R} - \mathbb{Q}} = (\mathbb{R} - \mathbb{Q}) \cup (\mathbb{R} - \mathbb{Q})' = (\mathbb{R} - \mathbb{Q}) \cup \mathbb{R} = \mathbb{R}$.

(20.5) Let Us Sum Up: *In a metric space* (X, d)*, we could define the notion of interior, exterior, frontier and boundary points of a subset. In this lesson, we have explicitly computed these for some subsets of a usual metric space* $(\mathbb{R}, d)$*.*

(20.6) Lesson end exercise

1. *Find the interior of* $[a, b]$ *in usual metric space* $(\mathbb{R}, d)$*.*

2. *Find the derived set of the following subsets of* $\mathbb{R}$ *in usual metric space:*

$(i) = (0, 1)$ $\quad (ii)$ $(0, 1]$ $\quad (iii)$ $[0, 1)$ $\quad (iv)$ $[0, 1]$.

3. *Find the derived set of the following subsets of* $\mathbb{R}$ *in usual metric space:*

(i) *singleton set* (ii) *finite subset of* $\mathbb{R}$

(iii) $\mathbb{N}$ $\quad (iv)$ $\mathbb{Z}$ $\quad (v)$ $\mathbb{Q}$ (vi) $\mathbb{R} - \mathbb{Q}$.

(20.7) University Model Questions

1. *Give an example to show that*

$$A^o \cup B^o \neq (A \cup B)^o.$$

Hint: *Take $A = \mathbb{Q}$ and $B = Ir$, the set of irrational numbers. Then $A^o = \phi$, $B^o = \phi$ and $(A \cup B)^o = \mathbb{R}$.*

2. *Show that the closure of open sphere is contained in the corresponding closed sphere. Also give an example to show that the closure of an open shpere is not necessarily a closed shere.*

3. *Give an example to show that*

$$\overline{A \cup B} \neq \overline{A} \cup \overline{B}.$$

Hint: *Take $A = \mathbb{Q}$ and $B = Ir$, the set of irrational numbers. Then $\overline{\mathbb{Q}} = \mathbb{R}$ and $\overline{Ir} = \mathbb{R}$. But $A \cap B = \phi$ and $\overline{A \cap B} = \phi$ and $\overline{\mathbb{Q}} \cap \overline{Ir} = \mathbb{R}$.*

4. *Prove that (i) $\overline{A^c} = (A^o)^c$*

(ii) $\overline{(A^c)} = (A^o)^c$

(iii) $b(A) = \overline{A} \cap \overline{A^c}$

(iv) $b(A) = \overline{A} - A^o$.

(20.8) Suggested Readings: *Shanti Narayanan, M. D. Raisinghania; Elements of Real Analysis, S. Chand and Company Pvt. Ltd Ramnagar New Delhi-110055.*

Lesson-XXI Continuous functions on metric spaces

(21.1) Introduction: *As we are familiar with the concept of continuity of a function on real numbers. Similarily we can study the concept of continuous functions on the metric spaces. In this lesson we will explain the properties of continuous functions on a metric space* (X, d).

(21.2) Objective: *The students will learn the continuity of functions on a metric space which is the generalisation of ral valued continuous functions on any metric space.*

(21.3) Continuous functions on metric spaces

(21.3.1) Definition: *Let* (X, d) *and* (Y, ρ) *be any two metric spaces. Then a function* $f : X \to Y$ *is said to be continuous at a point* $a \in X$ *if for every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that whenever*

$$d(x, a) < \delta \Rightarrow \rho(f(x), \epsilon).$$

In other words, for each open sphere $S(f(a), \epsilon)$ *centered at* $f(a)$, *there exists an open sphere* $S(a, \delta)$ *centered at* a *such that*

$$f(S(a, \delta)) \subset S(f(a), \epsilon).$$

Note: *A function* $f : X \to Y$*, which is continuous at every point of* X *is called continuous function.*

(21.3.2) Theorem. *Let* (X, d) *and* (Y, ρ) *be any two metric spaces. Then a function* $f : X \to Y$ *is continuous if and only if for each open subset* $G \subset Y$, $f^{-1}(G)$ *is open subset of* X.

Proof. *First, we suppose that* $f : X \to Y$ *is a continuous function and* G *is an open subset of* Y*. Then we will show that* $f^{-1}(G)$ *is open subset of* X*. For this, if* $f^{-1}(G) = \phi$*, then there is nothing to prove. Now, let* $f^{-1}(G) \neq \phi$ *and* $x \in f^{-1}(G)$*. Then* $f(x) \in G$.

Since G *is open set, so there exists* $\epsilon > 0$ *such that*

$$S_\rho(f(x), \epsilon) \subset G.$$

Also f *is continuous*

$\Rightarrow$ *there exists an open sphere* $S_d(x, \delta)$ *centered at* x *such that* $f(S_d(x, \delta)) \subset S_\rho(f(x), \epsilon) \subset G \Rightarrow S_d(x, \delta) \subset f^{-1}(G)$.

Thus for each $x \in f^{-1}(G)$*, there exists an open sphere* $S_d(x, \delta)$ *centered at* x *such that* $S_d(x, \delta) \subset f^{-1}(G)$*. Hence,* $f^{-1}(G)$ *is open in* X.

Conversely, suppose that for each open subset $G \subset Y$, $f^{-1}(G)$ *is open subset of* X*. Claim:* f *is continuous. For this, let* $x \in X$ *be any point. Then* $f(x) \in Y \Rightarrow$ *there exists an open sphere* $S_\rho(f(x), \epsilon)$ *centered at* $f(x)$ *in* Y.

Since every open sphere is an open set. Therefore, $S_\rho(f(x), \epsilon)$ *is an open subset of* Y*. Then by given condition,* $f^{-1}(S_\rho(f(x), \epsilon))$ *is open set in* X *and it contains* x.

Therefore, there exists an open sphere $S_d(x, \delta)$ *centere at* x *in* X *such that*

$$S_d(x, \delta) \subset f^{-1}(S_\rho(f(x), \epsilon))$$

$\Rightarrow f(S_d(x, \delta)) \subset S_\rho(f(x), \epsilon)$*. This shows that* f *is continuous at* x*, but* x *was* *an arbitrary point of X. Thus, f is continuous at every point of X and so f is continuous function.*

(21.3.3) Theorem. *Let (X, d) and (Y, ρ) be any two metric spaces. Then a function $f : X \to Y$ is continuous if and only if for each closed subset $K \subset Y$, $f^{-1}(K)$ is closed subset of X.*

Proof. *First, we suppose that $f : X \to Y$ is a continuous function and K is a closed subset of Y.*

Then we will show that $f^{-1}(K)$ is closed subset of X.

For this, we have K is a closed subset of Y,

then $Y - K$ is open subset of Y

$\Rightarrow f^{-1}(Y - K)$ *is open in X*

$\Rightarrow f^{-1}(Y) - f^{-1}(K)$ *is open in X*

$\Rightarrow f^{-1}(K)$ *is closed set in X.*

Conversely, suppose that $f : X \to Y$ is a function such that inverse image of every closed subset of Y is closed subset of X. We shall show that f is continuous.

For this, let G be an open set in $Y \Rightarrow Y - G$ is closed in Y

$\Rightarrow f^{-1}(Y - G)$ *is closed in X by given hypothesis*

$\Rightarrow f^{-1}(Y) - f^{-1}(G)$ *is closed in X*

$\Rightarrow X - f^{-1}(G)$ *is closed in X*

$\Rightarrow f^{-1}(G)$ *is open in X.*

Therefore, for each open set $G \subset Y$, we have $f^{-1}(G)$ is open in X. Thus f is continuous.

(21.3.4) Theorem. *Let (X, d) and (Y, ρ) be any two metric spaces. Then a function $f : X \to Y$ is continuous if and only if*

$$f(\overline{A}) \subset \overline{f(A)}, \forall A \subset X).$$

Proof. *First, we suppose that* f *is continuous function and let* $A \subset X$. *To show that*

$$f(\overline{A}) \subset \overline{f(A)}.$$

Note that $\overline{f(A)}$ *is a closed subset of* Y. *Since* f *is continuous, so* $\Rightarrow f^{-1}(\overline{f(A)})$ *is closed set in* X

$\Rightarrow \overline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)})$..........................(1)

Now, $f(A) \subset \overline{f(A)}$

$\Rightarrow A \subset f^{-1}(\overline{f(A)})$

$\Rightarrow \overline{A} \subset \overline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)})$ *(because of* (1))

$\Rightarrow f(A) \subset \overline{f(A)}$.

Conversely, suppose that

$$f(A) \subset \overline{f(A)}, \forall A \subset X.$$

To show that f *is continuous.*

For this, let K *be a closed subset of* $Y \Rightarrow \overline{K} = K$

Now, $f^{-1}(K)$ *is a subset of* X

therefore by given hypothesis $f(\overline{f^{-1}(K)}) \subset \overline{f(f^{-1}(K))} = \overline{K} = K$

i.e.$\overline{f^{-1}(K)} \subset f^{-1}(K)$ *but* $f^{-1}(K) \subset \overline{f^{-1}(K)}$

$\Rightarrow \overline{f^{-1}(K)} = f^{-1}(K)$

$\Rightarrow f^{-1}(K)$ *is closed set in* X

Therefore for all closed subset K *of* $Y \Rightarrow f^{-1}(K)$ *is closed set in* X. *Hence* f *is continuous function.*

(21.3.5) Theorem. *Let* (X, d) *and* (Y, ρ) *be any two metric spaces. Then a function* $f : X \to Y$ *is continuous if and only if*

$$\overline{f^{-1}(B)} \subset f^{-1}(\overline{B}), \forall B \subset Y.$$

Proof. *First, we suppose that f is continuous function and let $B \subset Y$. Then $\overline{B}$ is closed set in Y, since f is continuous,*
so $f^{-1}(\overline{B})$ is closed set in X
$\Rightarrow \overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B})$..........(1)
Now, $B \subset \overline{B} \Rightarrow f^{-1}(B) \subset f^{-1}(\overline{B})$
$\Rightarrow \overline{f^{-1}(B)} \subset \overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B})$ *(using* (1))
$\Rightarrow \overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.
Conversely, suppose that $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B}), \forall B \subset Y$. *To show that f is continuous.*
For this, let K be a closed subset of Y. Then by hypothesis, we have $\overline{f^{-1}(K)} \subset f^{-1}(\overline{K})$
$\Rightarrow \overline{f^{-1}(K)} \subset f^{-1}(\overline{K}) = f^{-1}(K)$
$\Rightarrow \overline{f^{-1}(K)} \subset f^{-1}(K)$
But $f^{-1}(K) \subset \overline{f^{-1}(K)}$
Therefore, $f^{-1}(K) = \overline{f^{-1}(K)} \Rightarrow f^{-1}(K)$ *is a closed subset of X. Thus for each closed subset K of $Y \Rightarrow f^{-1}(K)$ is closed subset of X. Hence f is continuous.*

(21.3.6) Theorem. *Let (X, d) and (Y, ρ) be any two metric spaces. Then a function $f : X \to Y$ is continuous if and only if*

$$f^{-1}(B^o) \subset \left\{f^{-1}(B)\right\}^o, \forall B \subset Y.$$

Proof. *First, we suppose that f is continuous function and let $B \subset Y$. Then B^o is open subset of Y, since f is continuous function. So $f^{-1}(B^o)$ is an open subset of X.*
$\Rightarrow (f^{-1}(B^o))^o = f^{-1}(B^o)$.................(1)
Now $B^o \subset B$
$\Rightarrow f^{-1}(B^o) \subset f^{-1}(B)$
$\Rightarrow (f^{-1}(B^o))^o \subset (f^{-1}(B))^o$ *use* (1)

$\Rightarrow f^{-1}(B^o) \subset (f^{-1}(B))^o$

Conversely, suppose that

$$f^{-1}(B^o) \subset \left\{f^{-1}(B)\right\}^o, \forall\, B \subset Y.$$

To show that f is continuous, let G be an open subset of Y

$\Rightarrow G^o = G.$

Therefore by the hypothesis $f^{-1}(G^o) \subset \{f^{-1}(G)\}^o$

$\Rightarrow f^{-1}(G) \subset \{f^{-1}(G)\}^o$

But $\{f^{-1}(G)\}^o \subset f^{-1}(G)$

$\Rightarrow \{f^{-1}(G)\}^o = f^{-1}(G)$

$\Rightarrow f^{-1}(G)$ *is open in* X.

Therefore, for all open set $G \subset Y \Rightarrow f^{-1}(G)$ *is open in* X. *Hence* f *is continuous.*

(21.4) Examples

1. *Let (X, d) be a metric space and x_0 be a fixed point of X. Show that the real valued function $f_{x_0}(x) = d(x, x_0)$ is continuous.*

Solution. *Let y be any point of X and $\epsilon > 0$ be any arbitrary real number. Then* $|f_{x_0}(x) - f_{x_0}(y)| = |d(x, x_0) - d(y, x_0)| \leq d(x, y)$ *(because* $d(x, A) - d(y, A) \leq d(x, y)$*)*

Now choose $\delta > 0$ such that $\delta \leq \epsilon$.

whenever $d(x, y) < \delta \Rightarrow |d(x, x_0) - d(y, x_0)| < \delta \leq \epsilon$.

Thus

$$d(x, y) < \delta \Rightarrow |f_{x_0}(x) - f_{x_0}(y)| < \epsilon.$$

This shows that f_{x_0} is continuous at an arbitrary point y, it foolows that f_{x_0} is continuous function.

2. *Let (X, d), (Y, ρ) and (Z, σ) be three metric spaces and $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Then prove that $gof : X \to Z$ is also*

continuous.

Solution. *Let G be an open set in Z. Then we have*

$$(gof)^{-1}(G) = (f^{-1}og^{-1})(G) = f^{-1}(g^{-1}(G))$$

Since g is continuous and G is open Z, so $g^{-1}(G)$ is open in Y. Now, since f is continuous and $g^{-1}(G$ is open in Y, it follows that $f^{-1}(g^{-1}(G))$ is open in X

$\Rightarrow (gof)^{-1}(G)$ *is open in X. Thus gof is continuous.*

(21.5) Let Us Sum UP*: In this lesson we have defined continuous function on metric spaces and then explained the various properties of continuous functions on metric spaces in the form of theorems.*

(21.6) Lesson End Exercise

1. *Define continuous function on metric spaces. Show that inverse image of closed set is closed set.*

2. *Let $(X,\, d)$ be a metric space and x_0 be a fixed point of X. Show that the real valued function $f_{x_0}(x) = d(x,\, x_0)$ is continuous.*

3. *Let $(X,\, d)$ be a metric space and S be a non-empty subset of X, then prove that the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x,\, S)\,\forall\, x \in X$ is continuous function.*

(21.7) Suggested Readings*: Shanti Narayanan, M. D. Raisinghania; Elements of Real Analysis, S. Chand and Company Pvt. Ltd Ramnagar New Delhi-110055.*

Lesson-XXII Convergent sequences in metric space

(22.1) Introduction: *Analogous to the notion of sequences of real numbers and their convergence, we shall study the sequences and their convergence in metric spaces. Further, we can investigate the properties of convergent sequences in metric spaces.*

(22.2) Objectives: *The students will learn the generalisation of convergence of sequences from set of real numbers to any metric space.*

(22.3) Sequences in a metric space

(22.3.1) Definition (Sequence): *Let (X, d) be a metric space. A function $s : \mathbb{N} \to X$ is called a sequence in a metric space. It is denoted by $\{s_n\}$, where s_n is called nth term of the sequence.*

For example $\{(-1)^n\}$ is sequence whose image has only two elements 1 and -1 whereas the sequence $\left\{\frac{1}{n}\right\}$ has infinite number of elements in its image.

(22.3.2) Definition (Subsequence): *A sequence* $\{t_n\}$ *is called a subsequence of the sequence* $\{s_n\}$ *if there exists a sequence of natural numbers* $\{n_k\}$ *such that* $n_1 < n_2 < n_3 < \ldots$ *and* $t_k = s_{n_k}$.

For example (i) $\{s_2,\, s_4,\, s_6,\, \ldots\}$ *is a subsequence of* $\{s_n | n \in \mathbb{N}\}$. *Here* $n_k = 2k$.

(ii) $\{s_1,\, s_4,\, s_9,\, \ldots\}$ *is a subsequence of* $\{s_n | n \in \mathbb{N}\}$. *Here* $n_k = k^2$.

(22.3.3) Definition (Convergent sequence): *A sequence* $\{s_n\}$ *is said to converge to a point* $s \in X$ *if for given* $\epsilon > 0$, *there exists* $m \in \mathbb{N}$ *such that* $d(s_n,\, s) < \epsilon\,,\, \forall\, n \geq m$.

In other words, the sequence $\{s_n\}$ *is said to converge to a point* $s \in X$ *if for given* $\epsilon > 0$, *there exists* $m \in \mathbb{N}$ *such that* $s_n \in S(s,\, \epsilon)\,,\, \forall\, n \geq m$.

Note: *If* $\{s_n\}$ *converges to* s, *we say that* s *is a limit of the sequence and we write*

$$\lim_{n\to\infty} s_n = s \text{ or } s_n \to s \text{ as } n \to \infty.$$

(22.3.4) Theorem. *Limit of the sequence* $\{s_n\}$, *if exists is unique.*

Proof. *Suppose that the sequence* $\{s_n\}$ *converges to two distinct points say* s *and* t. *Let* $r = d(s,\, t)$. *Then the open spheres* $S(s,\, \frac{r}{4})$ *and* $S(t,\, \frac{r}{4})$ *are disjoint. Since*

$$\lim_{n\to\infty} s_n = s$$

so there exists $m_1 \in \mathbb{N}$ *such that* $s_n \in S(s,\, \frac{r}{4})\,, \forall\, n \geq m_1$..........(1)

Similarily,

$$\lim_{n\to\infty} s_n = t$$

so there exists $m_2 \in \mathbb{N}$ *such that* $s_n \in S(t,\, \frac{r}{4})\,, \forall\, n \geq m_2$..........(2)

Choose $m = \max\{m_1,\, m_2\}$. *Then from* (1) *and* (2) *we have* $s_n \in S(s,\, \frac{r}{4})\,, \forall\, n \geq m$ *and* $s_n \in S(s,\, \frac{r}{4})\,, \forall\, n \geq m$

$\Rightarrow$ $S(s,\, \frac{r}{4}) \cap S(t,\, \frac{r}{4}) \neq \phi$ *which is a contradiction. Hence the limit* $\{s_n\}$ *is unique.*

(22.3.5) Theorem. *Let (X, d) and (Y, ρ) be two metric spaces. Then a function $f : X \to Y$ is continuous at $a \in X$ if and only if for each sequence $\{a_n\}$ in X converging to a, the sequence $\{f(a_n)\}$ converges to $f(a)$.*

Proof. *First, we suppose that $f : X \to Y$ is continuous at a and the sequence $\{a_n\}$ converges to a. Let $\epsilon > 0$. Since f is continuous at a, so there exists $\delta > 0$ such that*

when $d(x, a) < \delta \Rightarrow \rho(f(x), f(a)) < \epsilon$.............(1)

Also

$$\lim_{n\to\infty} a_n = a$$

so there exists $m \in \mathbb{N}$ such that

$$d(a_n, a) < \delta, \; \forall\, n \geq m............(2)$$

Put $x = a_n$ in (1), we get

$$d(x_n, a) < \delta \Rightarrow \rho(f(x_n), f(a)) < \epsilon.............(3)$$

From (2) and (3) we get $\rho(f(x_n), f(a)) < \epsilon, \forall\, n \geq m$

$\Rightarrow \{f(a_n)\} \to f(a)$ as $n \to \infty$.

Conversely, if possible, suppose that f is not continuous. Then we shall show that there exists a sequence $\{a_n\}$ converging to a but the sequence $\{f(a_n)\}$ does not converge to $f(a)$.

For this, since f is not continuous, so there must exist atleast one $\epsilon > 0$ such that for each $\delta > 0$ and for some $x \in X$,

$$d(x, a) < \delta \text{ and } \rho(f(x), f(a) \geq \epsilon).......(4)$$

Take $\delta = \frac{1}{n}$ in (4), we get, for each $n \in \mathbb{N}$, there exists $a_n \in X$ such that

$$d(a_n, a) < \frac{1}{n} \text{ but } \rho(f(a_n), f(a)) \geq \epsilon$$

$\Rightarrow \{f(a_n)\}$ *cannot converge to* $f(a)$.

(22.3.6) Theorem. *Let* x *and* y *be any two points in a metric space* (X, d) *and* $\{y_n\}$ *be a sequence converging to* y. *Then* $\{d(x, y_n)\}$ *converges to* $d(x, y)$.

Proof. *Since* $\{y_n\}$ *converges to* y

therefore, for given $\epsilon > 0$, *there exists* $m \in N$ *such that*

$$d(y_n, y) < \epsilon.........(1)$$

Now $|d(x, y_n) - d(x, y)| \leq d(y_n, y)$ *(because of Example* (21.4)(1)*)*

Using (1), *we get* $|d(x, y_n) - d(x, y)| \leq d(y_n, y) < \epsilon, \forall\, n \geq m$

$\Rightarrow d(x, y_n) \to d(x, y)$.

(22.3.7) Theorem. *Let* (X, d) *be a metric space and* $\{x_n\}$ *and* $\{y_n\}$ *be sequences in* X *such that* $x_n \to x$ *and* $y_n \to y$. *Then*

$$d(x_n, y_n) \to d(x, y).$$

Proof. *Since* $x_n \to x$ *and* $y_n \to y$, *so by definition there exist* $m_1, m_2 \in \mathbb{N}$ *such that*

$$d(x_n, x) < \epsilon/2, \forall\, n \geq m_1$$

and

$$d(y_n, y) < \epsilon/2, \forall\, n \geq m_2$$

Choose $m = \max\{m_1, m_2\}$. *Then* $d(x_n, x) < \epsilon/2, \forall\, n \geq m$ *and* $d(y_n, y) < \epsilon/2, \forall\, n \geq m$. *Now*

$$\begin{aligned} |d(x_n, y_n) - d(x, y)| &= |d(x_n, y_n) - d(x_n, y) + d(x_n, y) - d(x, y)| \\ &\leq |d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x, y)| \\ &\leq |d(y_n, y)| + |d(x_n, y)| < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

$\Rightarrow |d(x_n, y_n) - d(x, y)| < \epsilon, \forall\, n \geq m$

$\Rightarrow d(x_n, y_n) \to d(x, y)$.

(22.3.8) Theorem. *Let* (X, d) *be a metric space and let* $A \subset X$*. If* $x \in \overline{A}$*, then there exists a sequence* $\{x_n\}$ *of points of* A *such that* $x_n \to x$*.*

Proof. *Let* $x \in \overline{A}$*. Then* x *is an adherent point of* A

$\Rightarrow$ *for each* $r > 0$, $S(x, r) \cap A \neq \phi$.

Therefore, for each positive integer n*, the open sphere* $S(x, \frac{1}{n})$ *must contain a point* x_n *of* A *i.e.*$x_n \in S(x, \frac{1}{n})$

Now, we claim that $x_n \to x$ *as* $n \to \infty$*. For this, by Archimediean Property, for given* $\epsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that* $n_0\epsilon > 1$

$\Rightarrow \frac{1}{n_0} < \epsilon$*, therefore, for each* $n \geq n_0$ *we have* $\frac{1}{n} < \frac{1}{n_0} < \epsilon$

$\Rightarrow S(x, \frac{1}{n}) \subset S(x, \frac{1}{n_0}) \subset S(x, \epsilon)$

$\Rightarrow x_n \in S(x, \frac{1}{n}) \subset S(x, \frac{1}{n_0}) \subset S(x, \epsilon), \forall n \geq n_0$

$\Rightarrow x_n \in S(x, \frac{1}{n}) \subset S(x, \epsilon), \forall n \geq n_0$

$\Rightarrow x_n \in S(x, \epsilon), \forall n \geq n_0$

$\Rightarrow d(x_n, x) < \epsilon, \forall n \geq n_0.$

Hence $x_n \to x$ *as* $n \to \infty$*.*

(22.3.9) Theorem. *Let* (X, d) *be a metric space and* $A \subset X$*. If* $x \in A'$*, then there exists a sequence* $\{x_n\}$ *of points of* A *distinct from* x *which converges to* x*.*

Proof. *Let* $x \in A'$*. Then* x *is a limit point of* A

$\Rightarrow$ *each open sphere* $S(x, r)$ *centered at* x *contains atleast one point of* A *other than* x *i.e* $S(x, r) \cap A - \{x\} \neq \phi$.

Let $x_1 \neq x$ *such that* $x_1 \in A$ *and* $d(x, x_1) < r$

Let $r_1 = \min\{1, d(x, x_1)\}$

therefore, the open sphere $S(x, r_1)$ *contains atleast one point of* A *other than* x *i.e* $S(x, r_1) \cap A - \{x\} \neq \phi$.

Let $x_2 \neq x$ *such that* $x_2 \in A$ *and* $d(x, x_2) < r_1$

Let $r_2 = \min\{\frac{1}{2},\, d(x,\, x_2)\}$

Again, the open sphere $S(x,\, r_2)$ *contains atleast one point of* A *other than* x *i.e* $S(x,\, r_2) \cap A - \{x\} \neq \phi$.

Let $x_3 \neq x$ *such that* $x_3 \in A$ *and* $d(x,\, x_3) < r_2$

Let $r_3 = \min\{\frac{1}{3},\, d(x,\, x_3)\}$ *and continuing as above, we get a sequence* $\{x_n\}$ *of distinct points different from* x *such that*

$$r_n = \min\left\{\frac{1}{n},\, d(x,\, x_n)\right\}$$

and open sphere $S(x,\, r_n)$ *contains a point* x_{n+1} *of* A *other than* x *i.e* $S(x,\, r_{n-1}) \cap A - \{x\} \neq \phi$.

Thus $d(x,\, x_n) < r_{n-1} \leq \frac{1}{n-1}$

Taking limit as $n \to \infty$, *we get* $d(x,\, x_n) \to 0$ *and so* $x_n \to x$.

(22.4) Definition*: A sequence* $\{x_n\}$ *in a metric space* $(X,\, d)$ *is said to be a Cauchy sequence if for each* $\epsilon > 0$ *there exists a positive integer* n_0*(depending upon* ϵ*) such that*

$$d(x_n,\, x_m) < \epsilon,\, \forall\, n,\, m \geq n_0.$$

(22.4.1) Theorem. *Every convergent sequence in a metric space is a Cauchy sequence.*

Proof. *Let* $\{x_n\}$ *be a convergent sequence such that* $x_n \to x$ *in a metric space* $(X,\, d)$. *Then for given* $\epsilon > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that*

$$d(x_n,\, x) < \epsilon/2,\ \forall\, n \geq n_0.$$

Let $m \geq n_0$, *then* $d(x_m,\, x) < \epsilon/2,\, \forall\, m \geq n_0$.

Now $d(x_n,\, x_m) \leq d(x_n,\, x) + d(x,\, x_m),\, \forall\, n,\, m \geq n_0$

$< \epsilon/2 + \epsilon/2 = \epsilon$

$\Rightarrow d(x_n,\, x_m) < \epsilon,\, \forall\, n,\, m \geq n_0$. *Hence* $\{x_n\}$ *is a Cauchy sequence.*

Note: *The converse of this theorem is not necessarily true. i.e a Cauchy sequence need not be convergent sequence in a metric space.*

For example, let $X = (0, 1]$ *be a usual metric space with metric* d*. Let* $\{x_n\}$ *be a sequence in* X *such that* $x_n = \frac{1}{n}, \forall\, n \in \mathbb{N}$*. We shall show that* $\{x_n\}$ *is a Cauchy sequence. For this, since for given* $\epsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that* $n_0 > \frac{2}{\epsilon}$ *(by Archimedean Property), then*

for $n,\, m \geq n_0 \Rightarrow \frac{1}{n} \leq \frac{1}{n_0} < \frac{\epsilon}{2}$ *and* $\frac{1}{m} \leq \frac{1}{n_0} < \frac{\epsilon}{2}$

Now $d(x_n,\, x_m) = |x_n - x_m| \leq |x_n| + |x_m|$

$= \frac{1}{n} + \frac{1}{m}$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$

Thus, $d(x_n,\, x_m) < \epsilon, \forall\, n,\, m \geq n_0$

Therefore $\{x_n\}$ *is a Cauchy sequence.*

But

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0 \notin X.$$

Hence a Cauchy sequence need not converge to any point of the metric space.

(22.4.2) Theorem. *A Cauchy sequence* $\{x_n\}$ *is convergent in a metric space if and only if it has a convergent subsequence.*

Proof. *Firstly, let us suppose that* $\{x_n\}$ *is Cauchy sequence and* $\{x_{n_k}\}$ *be a subsequence of* $\{x_n\}$ *converging to* $x \in X$*. Then we shall show that* $\{x_n\}$ *is convergent sequence. For this, since every convergent sequence is Cauchy so* $\{x_{n_k}\}$ *is also Cauchy sequence.*

Therefore, for given $\epsilon > 0$*, there exists a positive integer* m *such that*

$$d(x_{n_k},\, x_n) < \epsilon/2,\, \forall\, k,\, n \geq m.....(1)$$

Also $\{x_{n_k}\} \to x$

$\Rightarrow d(x_{n_k},\, x) < \epsilon/2, \forall\, k \geq p$ *for some* $p \in \mathbb{N}$*..........(2).*

Let $k = m + p = r$*, say. Then* (2) *becomes*

$d(x_{n_r}, x) < \epsilon/2$.........(3).

Also note that $n_r \geq r \geq q$ and hence (1) becomes

$$d(x_{n_r}, x_n) < \epsilon/2, \forall\, r,\, n \geq m.....(4)$$

Now by tiangle inequality $d(x, x_n) \leq d(x, x_{n_k}) + d(x_{n_k}, x_n)$

$$< \epsilon/2 + \epsilon/2 = \epsilon$$

$\Rightarrow d(x, x_n) < \epsilon, \forall\, n \geq m.$

Hence $\{x_n\}$ is convergent.

Conversely, suppose that $\{x_n\}$ is a Cauchy sequence which is also convergent in a metric space (X, d). To show that $\{x_n\}$ has a convergent subsequence. For this, since $\{x_n\}$ is convergent, suppose it converges to $x \in X$. Then the constant sequence $\{x, x, \ldots\}$ is a subsequence which is also convergent.

(22.5) Examples

1. *Let $\{s_n\}$ be a Cauchy sequence in a metric space (X, d) and $\{s_{n_i}\}$ be a subsequence of $\{s_n\}$, then show that*

$$\lim_{n\to\infty} d(s_{n_i}, s_n) = 0.$$

Solution. *Since $\{s_n\}$ is a Cauchy sequence, so for given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that*

$$d(s_m, s_n) < \epsilon, \forall\, m,\, n \geq n_0$$

Consider $\{n_i\}$ such that $n_0 \leq n \leq n_1 \leq n_2 \leq \ldots$. Then

$$d(s_{n_i}, s_n) < \epsilon, \forall\, n,\, n_i \geq n_0.$$

Hence we have

$$\lim_{n\to\infty} d(s_{n_i}, s_n) = 0.$$

2. *Let $\{b_n\}$ be a Cauchy sequence in a metric space (X, d) and let $\{a_n\}$ be a sequence in X such that $d(a_n, b_n) < 1/n$ for every $n \in \mathbb{N}$ then show that $\{a_n\}$*

is a Cauchy sequence in X.

Solution. *Since $\{b_n\}$ is a Cauchy sequence, so for given $\epsilon > 0$ there exist $m_1 \in \mathbb{N}$ such that*

$$d(b_m, b_n) < \epsilon/3, \forall\, m, n \geq m_1.........(1).$$

Now, by Archimedian property, for ϵ, we can find positive integer m_2 such that $\frac{1}{m} < \epsilon/3$ and $\frac{1}{n} < \epsilon/3$, $\forall\, m, n \geq m_2$.
Choose $m_0 = \max\{m_1, m_2\}$,
then $m, n \geq m_0 \Rightarrow d(b_m, b_n) < \epsilon/3$, $\frac{1}{m} < \epsilon/3$, $\frac{1}{n} < \epsilon/3........(*)$
Now, by triangle inequality, for $m, n \geq m_0$ we have

$$\begin{aligned} d(a_m, a_n) &\leq d(a_m, b_m) + d(b_m, b_n) + d(b_n, a_n) \\ &< \tfrac{1}{m} + \epsilon/3 + \tfrac{1}{n} \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

$\Rightarrow d(a_m, a_n) < \epsilon$, $\forall\, m, n \geq n_0$. Hence $\{a_n\}$ is a Cauchy sequence.

(22.6) Let Us Sum Up: *In this lesson, we have defined the notion of sequence, subsequence and the convergence of a sequence in a metric space. Then we have explained the properties of convergent sequences in a metric space via theorems and examples.*

(22.7) Lesson End Exercise

1. *If $\{x_n\}$ is a Cauchy sequence in some metric space (X, d), and a subsequence $\{x_{n_k}\}$ converges to a limit $x \in X$, show that $\{x_n\}$ converges to x.*
2. *Show that a subsequence of a Cauchy sequence must be a Cauchy sequence.*
3. *If $\{x_n\}$ and $\{y_n\}$ are sequences in a space with metric d such that $\{x_n\}$ is a Cauchy sequence and $d(x_n, y_n) \to 0$, show that $\{y_n\}$ is also a Cauchy sequence.*

(21.7) Suggested Readings:(20.8) Suggested Readings: *Shanti Narayanan, M. D. Raisinghania; Elements of Real Analysis, S. Chand and Company Pvt. Ltd Ramnagar New Delhi-110055.*